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Life Insurance and Pension Contracts I: The Time Additive Life Cycle Model

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Abstract

We analyze optimal consumption in the life cycle model by introducing life and pension insurance contracts. The model contains a credit market with biometric risk, and market risk via risky securities. This idealized framework enables us to clarify important aspects life insurance and pension contracts. We find optimal pension plans and life insurance contracts where the benefits are state dependent. We compare these solutions both to the ones of standard actuarial theory, and to policies offered in practice. Implications of this include what role the insurance industry may play to improve welfare. The relationship between substitution of consumption and risk aversion is highlighted in the presence of a consumption puzzle. One problem related portfolio choice is discussed - the horizon problem. Finally, we present some comments on longevity risk and cohort risk.

KEYWORDS: The life cycle model, pension insurance, optimal life insurance, longevity risk, the horizon problem, consumption puzzle.
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1 Introduction

Four or five decennials back life and pension insurance seemed less problematic than today, at least from the insurance companies' point of view. Prices were set by actuaries using life tables, and a "fixed calculation" interest rate. This rate was not directly linked to the equilibrium interest rate of the market, or any other market linked quantities or indexes. The premium reserves

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of the individual and collective policies were invested in various assets, and when the different contracts were settled, the evolution of the premium reserve determined the final insurance compensation. If the return on the premium reserve had been higher than the calculation rate, this gave rise to a bonus. For a mutual company "bonus" need not only involve a payment from the insurer to the customer, but could also involve a payment in the other direction. For a stock owned corporation the bonus could in principle only be non-negative. In most cases this did not matter all that much, since the calculation rate was set to the safe side, which meant lower than the realized return rate on the premium reserve.

In several countries the nominal interest rate was high during some parts of this period, often significantly higher than the fixed rate used in determining premiums. In Norway, for example, this calculation rate (4%) appeared from some point in time as a legal guaranteed return rate in the contracts. For current policies this guarantee is reduced to 3%, and even lower.

During the last two or three decennials this interest rate guarantee has become a major issue for many life insurance companies. What initially appeared to be a benefit with almost no value, later turned out to be rather valuable for the policy holders, and correspondingly problematic for the insurers.

In this paper we study demand theory under idealized conditions using the life cycle model. This model takes the security market as given. Although the model does not explicitly contain insurance companies, nor a public sector, optimal insurance contracts are assumed to exist. We derive optimal contracts in this complete model, and compare these to contracts that are offered in the real world. We argue that the insurance industry can provide more consumption substitution over the life time of the consumers, than they can manage alone - since companies do not have any finite horizon. The preference structure implies that some smoothing in consumption is desired by the individuals. However, the analysis reveals a consumption puzzle when confronted with aggregate consumption and market data. Among other things this says that the representative individual do not prefer quite as much smoothing as implied by the real data. Thus something seems to be wrong with the model employed for the individual - it does not quite match reality. Recursive utility, the subject of a companion paper, give better results on these particular issues.

Every downturn in the financial market has typically been accompanied by problems for the life insurance industry. This is particularly true for privately owned life insurance companies, with a regulatory regime that focuses on yearly results. Collective pension funds with a different form of regulation seem less affected. For both types of companies the contracts offered are typically long term.

In view of this, managements of privately owned life insurance companies seem to prefer to offer "defined contribution" type policies to the more traditional "defined benefit" ones. The former type exposes the companies less to risk than the latter, and equity can be set lower. A thought provoking observation is then that when customers are asked what type of contract they prefer, the answer is typically defined benefit, i.e., the contract with most consumption smoothing. This is consistent with the view that customers can, by and large, manage ordinary savings themselves, including saving through the financial markets, for example by investing in mutual funds. What they need from an insurer is precisely - insurance. This means a reliable arrangement providing yearly payment of known real value to cover subsistence, and a little more, in the case that the individual's savings strategy did not work out all that well.

One would think that life and pension insurance companies should be able to offer precisely this kind of insured pensions to the public. This industry is normally perceived as taking a long term perspective, and should be able to let the equity premium work to their advantage in the long run. While an individual customer may have problems to carry out an optimal substitution of consumption during his/her life time because of a bounded life span, the insurance industry is presumably less constrained in this regard, and should be able to "time diversity".

If the insurance industry only offers defined contribution, or unit linked-type pension plans, finance theory tells us that the industry can only expect to earn the risk-free rate in the long run. The insurers will then compete with investment funds and other financial intermediaries, and the fees should eventually come down due to competition. The resulting return on the insurance companies' operations is unlikely to meet the requirements of the owners. On the other hand, there is the principle of *dynamic consistency*, which tells us that when there is some product demanded by enough people, there will eventually be a market for this product. So where does that lead the insurance industry? These are some of the topics discussed in this paper.

When there is consumption in several periods in a world with a perfect credit market with no financial risk, the standard model turns out to works just fine. This is also the case in a one period problem with financial uncertainty, a so-called *timeless* situation. When there is consumption in several periods (at least two) and there is also financial risk, we have a so-called *temporal* problem. In such situations induced preferences may not satisfy the substitution axiom, so the von Neumann-Morgenstern expected utility (Eu) theory does not have axiomatic underpinnings. This problem is taken up in the companion paper.

The paper is organized as follows: In Section 2 we introduce consumption and saving with only a credit market available. Here we explain some actuarial concepts related to mortality. In particular we study the effects from pooling.

In Section 3 we include mortality risk, i.e., an uncertain planning horizon, in the model of Section 2 and derive both optimal life insurance as well as optimal pension insurance, and investigate their properties when there is only a credit market present.

In Section 4 we introduce a financial market for risky securities in addition to the pure risk free credit market. Here we derive the optimal consumption and pension insurance, and show that with pension insurance available, the actual consumption rate at each time is larger than without.

In Section 5 we discuss a consumption puzzle, when the theory is confronted with real data. In Section 6 we discuss business cycles. In Section 7 we derive implications of the optimal pension plan, and discuss comparative statics. In Section 8 the connection to actuarial theory and insurance practice is briefly taken up, and in Section 9 a one-period "timeless" model is presented in order to analyze to what extent a pension insurance works as diversification. In Section 10 we finally analyze life insurance. Here we determine the *optimal amount* of life insurance, a state dependent quantity - a result we claim is new, and discuss its possible relevance to the insurance industry. The portfolio choice problem is briefly studied in Section 11, in Section 12 we point out a solution to *time horizon problem* and in Section 13 we study a second portfolio choice puzzle. Finally, in Section 14 we reflect on longevity risk and cohort risk, and give in Section 15 a summary discussion of our results, where we also suggest some extensions.

2 Consumption and Saving

In our development it will be an advantage to start with the simplest problem in optimal demand theory, when there is no risk and no uncertainty of any kind.

Consider a person having income $e(t)$ and consumption $c(t)$ at time t. Given income, possible consumption plans must depend on the possibilities for saving and for borrowing and lending. We want to investigate the possibilities of using income during one period to generate consumption in another period.

To start, assume the consumer can borrow and lend to the same interest rate r. Given any e and c, the consumer's net saving $W(t)$ at time t is

$$W(t) = \int_0^t e^{r(t-s)}(e_s - c_s)ds. \tag{1}$$

Assuming the person wants to consume as much as possible for any e, not any consumption plan is feasible. A constraint of the type $W(t) \geq a(t)$ may seem reasonable: If $a(t) < 0$ for some t, the consumer is allowed a net debt at time t. Another constraint could be $W(T) \geq B \geq 0$, where T is the planner's horizon. The consumer is then required to be solvent at time T.

The objective is to optimize the utility $U(c)$ of lifetime consumption subject to a budget constraint. There could also be a bequest motive when life insurance is an issue.

2.1 Uncertain planning horizon

In order to formulate the most natural budget constraint of an individual, which takes into account the advantages of pooling risk, we introduce mortality. Yaari (1965), Hakansson (1969) and Fisher (1973) were of the first to include an uncertain lifetime into the theory of the consumer.

The remaining lifetime T_x of an x year old consumer at time zero is a random variable with support $(0, \tau)$ and cumulative probability distribution function $F^x(t) = P(T_x \leq t)$, $t \geq 0$. The survival function is denoted by $\bar{F}^x(t) = P(T_x > t)$. Ignoring possible selection effects, it can be shown that

$$\bar{F}^x(t) = \frac{l(x+t)}{l(x)} \tag{2}$$

for some function $l(\cdot)$ of one variable only. The decrement function $l(x)$ can be interpreted as the expected number alive in age x from a population of $l(0)$ newborne.

The force of mortality, or death intensity, is defined as

$$\mu_x(t) = \frac{f_x(t)}{1 - F^x(t)} = -\frac{d}{dt} \ln \bar{F}^x(t), \qquad F^x(t) < 1, \tag{3}$$

where $f_x(t)$ is the probability density function of T_x. Integrating this expression yields the survival function in terms of the force of mortality

$$\bar{F}^x(t) = \frac{l(x+t)}{l(x)} = \exp\Big\{ -\int_0^t \mu_x(u)\, du \Big\}. \tag{4}$$

Suppose $y \geq 0$ a.s. is a non-negative process in L, the set of consumption processes. Later L will be a set of adapted stochastic processes y satisfying $E(\int_0^\tau y_t^2 dt) < \infty$. If T_x and y are independent, the formula

$$E\Big(\int_0^{T_x} y_t dt\Big) = \int_0^\tau E(y_t)\frac{l(x+t)}{l(x)}dt = \int_0^\tau E(y_t)e^{-\int_0^t \mu_x(u)du}dt \tag{5}$$

follows essentially from integration by parts, the independence assumption and Fubini's Theorem. Assuming the interest rate r is a constant, it follows that the single premium of an annuity paying one unit per unit of time is given by the actuarial formula

$$\bar{a}_x^{(r)} = \int_0^\tau e^{-rt}\,\frac{l_{x+t}}{l_x}dt, \tag{6}$$

and the single premium of a "temporary annuity" which terminates after time n is

$$\bar{a}_{x:\bar{n}|}^{(r)} = \int_0^n e^{-rt}\,\frac{l_{x+t}}{l_x}dt. \tag{7}$$

Under a typical pension plan the insured will pay a constant, or "level" premium p up to some time of retirement n, and from then on he will receive an annuity b as long as he lives. *The principle of equivalence* gives the following relationship between premium and benefit:

$$p\int_0^n e^{-rt}\,\frac{l_{x+t}}{l_x}dt = b\int_n^\tau e^{-rt}\,\frac{l_{x+t}}{l_x}dt.$$

In standard actuarial notation this is written

$$p\bar{a}_{x:\bar{n}|}^{(r)} = b(\bar{a}_x^{(r)} - \bar{a}_{x:\bar{n}|}^{(r)}). \tag{8}$$

The following formulas are sometimes useful in life insurance computations

$$\mu_x(t) = -\frac{l'(x+t)}{l(x+t)}, \qquad \text{and} \qquad f_x(t) = -\frac{l'(x+t)}{l(x)} = \frac{l(x+t)}{l(x)}\mu_{x+t}, \tag{9}$$

where $l'(x+t)$ is the derivative of $l(x+t)$ with respect to t. The present value of one unit payable at time of death is denoted $\bar{A}_x$. Using (9) and integration by parts, it can be written

$$\bar{A}_x = \int_0^\tau e^{-rt}f_x(t)dt = 1 - r\bar{a}_x^{(r)}. \tag{10}$$

This insurance contract is called *Whole life insurance.* If the premium rate p is paid until the retirement age n for a combined life insurance with z units payable upon death, and an annuity of rate b per time unit as long as the insured is alive, we have the following relationship between p, b and z:

$$p\bar{a}^{(r)}_{x:\bar{n}|} = b(\bar{a}^{(r)}_{x} - \bar{a}^{(r)}_{x:\bar{n}|}) + z(1 - r\bar{a}^{(r)}_{x}). \tag{11}$$

Pension insurance and life insurance can now be integrated in the life cycle model in a natural way, as we shall demonstrate.

2.2 The effect from pooling

In the discussion of consumption and saving, the following quantity plays an important role:

$$E\{W(T_x)e^{-rT_x}\} = \text{expected discounted net savings.} \tag{12}$$

In the absence of a life and pension insurance market, one would, as before, consider consumption plans c such that $W(T_x) \geq B$, or

$$W(T_x)e^{-rT_x} \geq b \geq 0 \quad \text{almost surely} \tag{13}$$

e.g., debt must be resolved before the time of death. If, on the other hand, pension insurance is possible, one can allow consumption plans where

$$E\{W(T_x)e^{-rT_x}\} = 0 \quad \text{(no life insurance.)} \tag{14}$$

Those individuals who live longer than average are guaranteed a pension as long as they live via the pension insurance market. The financing of this benefit comes from those who live shorter that average, which is what pooling is all about.

The implication is that the individual's savings possibilities are "exhausted", by allowing gambling on own life length. Clearly the above constraint in (14) is less demanding than requiring that the discounted net savings, the random variable in (13), is larger that some non-negative number b with certainty. Integration by parts gives the following expression for the expected discounted net savings

$$E\{W(T_x)e^{-rT_x}\} = \int_0^{\tau} \big(e(t) - c(t)\big)e^{-\int_0^t (r+\mu_{x+u})du}dt. \tag{15}$$

This expression we have interpreted as the present value of the consumer's net savings, which is seen from (15) to take place at a "spot" interest rate

$$r + \mu > r$$

where the inequality follows since the mortality rate $\mu > 0$. This is a result of the the pooling effect of (life and) pension insurance. The existence of a life and pension insurance market allows the individuals to save at a higher interest rate than the spot rate r. With a pure pension insurance contract, the policyholder can consume *more* while alive, since terminal debt is resolved by pooling. This is illustrated later in an example when also market uncertainty is taken into account.

Example 1. (A Pension Contract, or an Annuity). Suppose $e(t) = 0$ for $t > n$. The condition $E\big\{W(T_x)e^{-rT_x}\big\} = 0$ is seen to correspond to the *Principle of Equivalence* in this situation:

$$\int_0^n \big(e(t) - c(t)\big)P[T_x > t]e^{-rt}dt = \int_n^\tau c(t)e^{-rt}P[T_x > t]dt. \tag{16}$$

Here the difference $(e_t - c_t)$ can be interpreted as the premium (intesity) p_t paid while working, giving rise to the "pension" (or total consumption rate) c_t after the time of retirement n. This relationship implies that the pension is paid out to the beneficiary as long as necessary, and only then, i.e., as long as the policy holder is alive. $\square$

Notice the similarity between the actuarial formula in (8) and the above equation (16). Both equations are, of course, based on the same principle. It presents no difficulty to separate ordinary savings from pension in the above, but for the sake of simplicity of exposure, we shall employ an integrated approach in what follows.

3 The optimal demand theory with only a credit market

3.1 Introduction

In order to analyze the problem of optimal consumption (including optimal pension), we need assumptions about the preferences of the consumer. To start, we assume the preferences are represented by a utility function $U : L \to R$ given by the additive and separable von Neumann-Morgenstern expected utility of the form

$$U(c) = E\Big\{\int_0^{T_x} e^{-\delta t}u(c_t)dt + e^{-\kappa T_x}v(W_{T_x})\Big\}. \tag{17}$$

Here δ and κ represent utility discounting, and are interpreted as impatience rates, u is a strictly increasing and concave utility function, and v is a another utility function. The function v is connected to life insurance, and may represent a bequest motive, but as we will argue later, "bequest" is not always the most natural cause for life insurance. The functions u and v are sometimes referred to as felicity indices. Later, in Section 5, we refine this representation of preferences to recursive utility.

The classical reference to this material is of course Ramsey (1928). We could, in a natural way, have extended the analysis to include a recursive structure of preferences like in Koopmans (1960), which is often taken as a precursor to recursive utility. As it turns out, the standard model can manage well when there is no risk, so this would be to complicate things unnecessary. The possible problem with this model is simply that the world contains risk, the model does not.

The variable $z = W(T_x)$ is the amount of life insurance. It is often assumed to be a given constant (e.g., 1) in the standard theory of life insurance, but we will allow it to be a random variable that the decision maker can have some influence on. First we focus on pensions and annuities and set $v \equiv 0$.

3.2 The pension problem

The pension problem may then be formulated as:

$$\max_c E\Big\{ \int_0^{T_x} e^{-\delta t} u(c_t) dt \Big\} \tag{18}$$

subject to (i) $E\big(W(T_x)e^{-rT_x}\big) = 0$, and (ii) $c_t \geq 0$ for all t. Ignoring the positivity constraint (ii) for the moment, we may use Kuhn-Tucker and a variational argument to solve this problem. The Lagrangian is

$$\mathcal{L}(c;\lambda) = \int_0^{\tau} u(c_t) e^{-\int_0^t (\delta+\mu_{x+s})ds} dt + \lambda\Big(\int_0^{\tau} \big(e(t) - c(t)\big) e^{-\int_0^t (r+\mu_{x+s})ds} dt\Big).$$

If $c^*(t)$ is optimal, there exists a Lagrange multiplier λ such that $\mathcal{L}(c;\lambda)$ is maximized at $c^*(t)$ and complementary slackness holds. Denoting the directional derivative of $\mathcal{L}(c^*;\lambda)$ in the direction c by $\bigtriangledown\mathcal{L}(c^*,\lambda;c)$, the first order condition of this unconstrained problem is

$$\bigtriangledown\mathcal{L}(c^*,\lambda;c) = 0 \quad \text{in all 'directions' } c \in L,$$

which is equivalent to

$$\int_0^{\tau} \big(u'(c_t^*) e^{-\int_0^t (\delta+\mu_{x+s})ds} - \lambda e^{-\int_0^t (r+\mu_{x+s} ds}\big) c(t) dt = 0, \ \forall c \in L.$$

This gives the first order condition

$$u'(c_t^*) = \lambda e^{-(r-\delta)t}, \quad t \geq 0. \tag{19}$$

Notice that the force of mortality $\mu_x(t)$ does not enter this expression.

Differentiating this function in t along the optimal path c^*, we deduce the following differential equation for c^*

$$\frac{dc_t^*}{dt} = (r - \delta)T(c_t^*), \tag{20}$$

where T(c) $= -\frac{u'(c)}{u''(c)}$. When financial risk is present, this quantity is interpreted as the *absolute risk tolerance function* of the consumer, the reciprocal of the *absolute risk aversion function* A(c) i.e., T(c) = 1/A(c). Here this interpretation does not make much sense, since there is no (financial) risk, only biometric risk which we assume can be diversified away by pooling. The interpretation in the present situation is, perhaps, best illustrated by an example.

Exampel 2. (A Pension Contract for the CEIS Consumer.) Assume that the income process e_t is:

$$e_t = \begin{cases} y, & \text{if } t \leq n; \\ 0, & \text{if } t > n \end{cases} \tag{21}$$

where y is a constant, interpreted as the consumer's salary when working. The utility function is assumed power utility $u(c) = \frac{1}{1-\rho} c^{1-\rho}$. The parameter $\rho \geq 0$ is called the *time preference* parameter [1]. This utility function has a constant elasticity of intertemporal substitution (CEIS) in consumption, denoted by ψ and related to the parameter ρ via $\psi = \frac{1}{\rho}$.

Returning to the first order conditions, the optimal consumption (and pension) is then $c_t^* = k e^{\frac{1}{\rho}(r-\delta)t}$, where k is an integration constant. Equality in constraint (i) determines the constant k: The optimal life time consumption ($t \in [0, n]$) and pension ($t \in [n, \tau)$) is then

$$c_t^* = y \; \frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(r_0)}_x} \; e^{\frac{1}{\rho}(r-\delta)\,t} \quad \text{for all } t \geq 0. \tag{22}$$

Here $r_0 = r - \frac{r-\delta}{\rho}$ and $\bar{a}^{(r)}_{x:\bar{n}|}$ and $\bar{a}^{(r_0)}_x$ are the actuarial formulas explained in (6) and (7). Although the first order conditions in (19) do not depend on

[1] Some readers may now be confused and rename this parameter γ, and interpret it as the coefficient of relative risk aversion (CRRA). Again, this interpretation is meaningless here, since there is no financial risk.

mortality, the optimal consumption c_t^* does, since the Lagrange multiplier λ, or equivalently, the integration constant k, is determined from the 'average' budget constraint (i). Also, the positivity constraint (ii) is not binding at the optimum, due to the form of the felicity index u. Notice that in this example, T(x) = ψ x for all x. $\square$

For the CEIS-utility of this example, we notice that the function $A(x) = \rho/x$ is associated with the time preference ρ, and the function T is similarly associated to the EIS-parameter ψ.

3.3 The effects from changing EIS

The differential equation (20) tells us that the value of the interest rate r is a crucial border value for the impatience rate δ. When $\delta > r$ the optimal consumption c_t^* is always a decreasing function of time t, when $\delta < r$ the optimal consumption increases with time. In the first case, the 'impatient' one has already consumed so much, that he can only look forward to a decreasing consumption path. The 'patient' one can, on the other hand, look forward to a steadily increasing future consumption path. In Example 2 we see from (22) that the former has an optimal consumption path that is a decreasing exponentially, while the latter has an exponentially increasing consumption path. This seems to suggest that it may be difficult to compare consumption paths between different consumers. That this is not so clear-cut as this example might suggest, will follow when we introduce a securities market where the consumers are allowed to invest in risky securities as well as a risk-less asset in order to maximize lifetime consumption.

In Example 2 we can derive comparative statics for the EIS-parameter $\psi = \frac{1}{\rho}$. This can be inferred from the following

$$\frac{\partial}{\partial \psi} c_t^* = y \; \frac{\bar{a}^{(r)}_{x:\bar{n}|} e^{\psi\,(r-\delta)\,t}}{(\bar{a}^{(r_0)}_{x})^2} \, (r-\delta)\,(t-\tau_0)\,, \tag{23}$$

where the constant time τ_0 is found from the first mean value theorem for integrals:

$$\int_0^{\tau} s \; \frac{l_{x+s}}{l_x} \, e^{-r_0 s} \, ds = \tau_0 \int_0^{\tau} \frac{l_{x+s}}{l_x} \, e^{-r_0 s} \, ds$$

for some $\tau_0 \in (0, \tau)$. For the patient individual, an increase in ψ leads to an increase in consumption later on (i.e., for $t > \tau_0$) and a decrease earlier in life ($t < \tau_0$). For the impatient individual the conclusions are just the opposite.

3.4 The effects from changes in the interest rate

It is also of interest to explore the effect on optimal consumption of an increase in the interest rate. This will shed some further light on the interpretation of the EIS in the present situation. Loosely speaking, EIS deals with the individual's ability to manage deterministic variations in consumption in order to increase overall utility. In the present case with no financial risk, it is indeed the EIS interpretation, or equivalently, time preference, that is relevant. When risk is introduced, the parameter ρ will play more than one role for the conventional model.

An individual with a low value of ρ requires less compensation in the future for a decrease in consumption today, than an individual with a larger value of ρ. If an individual has a low value of ρ, this means that this person is relatively "neutral" to consumption substitution across time. The individual has a high ability to do this type of transfer, and, will need typically need little help from others, like a life and pension insurance company, or other financial institution. This person has an associated large value for ψ. When $\rho = 0$ the individual *is* neutral with respect to consumption substitution, and has an infinite EIS-parameter.

Think of a bear living in the northern hemisphere as having a large value of the EIS-parameter ψ. This animal may easily postpone consumption for several months, and is well-suited to tackle the significant deterministic variations in consumption posed by the differences between the seasons. A lemming, to take another example, could not postpone consumption in this way since it would then simply die.[2] The property of consumption substitution has nothing to do with risk aversion, which is addressing something else, namely the individual's attitude to variations across the *states of nature*.

Returning to the effect on the optimal consumption of an increase in the risk free interest rate r, we get

$$\frac{\partial}{\partial r} c_t^* = y \, \frac{\bar{a}_{x:\bar{n}|}^{(r)} e^{\frac{1}{\rho}(r-\delta)\,t}}{(\bar{a}_x^{(r_0)})^2} \, \psi \left(t - \Big(\frac{1}{\psi} \tau_1^{(n)} + (1 - \frac{1}{\psi}) \tau_2 \Big) \right), \tag{24}$$

where the two time points $\tau_1^{(n)}$ and τ_2 are defined by first mean value theorem for integrals as follows

$$\int_0^n s \, \frac{l_{x+s}}{l_x} \, e^{-rs} \, ds = \tau_1^{(n)} \int_0^n \frac{l_{x+s}}{l_x} \, e^{-rs} \, ds$$

and

$$\int_0^\tau s \, \frac{l_{x+s}}{l_x} \, e^{-r_0 s} \, ds = \tau_2 \int_0^\tau \frac{l_{x+s}}{l_x} \, e^{-r_0 s} \, ds$$

[2] On another time scale a lemming might have a larger EIS.

respectively. From the expression (24) we see that an increase in the interest rate r leads to an increase in the optimal consumption later, and a decrease in the optimal consumption earlier in the individual's life span, provided the break-point-in-time $\tilde{t} = \Big(\frac{1}{\psi}\tau_1^{(n)} + (1-\frac{1}{\psi})\tau_2\Big)$ is strictly positive. $\tilde{t}$ is seen to be a convex combination of the two time points $\tau_1^{(n)}$ and τ_2 when $\psi \geq 1$, and when $\psi = 1$, $\tilde{t} = \tau_1^{(n)}$. We conjecture that $\tau_1^{(n)} < \tau_2$ since the pension age $n < \tau$, but this also depends upon the relation between r and r_0. This means that $\tilde{t} = (\tau_2 - \frac{1}{\psi}\big(\tau_2 - \tau_1^{(n)}\big))$ is an increasing function of ψ, so that when ψ increases, $\tilde{t}$ approaches τ_2. An individual with a EIS-parameter $\psi \geq 1$ will use an increase in the interest rate to save more early when $t < \tilde{t}$, and accordingly consume more later when $t > \tilde{t}$.

When ψ is smaller than one, an increase in the interest rate will not necessarily have this substitution effect, and the "income effect" may dominate.

This is of course an important observation related to the insurance industry. According to this result will individuals with $\psi \geq 1$ react to an increase in the interest rate potentially different from an individual with $\psi < 1$.

This naturally leads to the question of how large the EIS-parameter is for the representative insurance customer. Below, but primarily in the companion paper we shall return to this question.

3.5 Including life insurance

It is quite natural to also study life insurance in this framework, where the goal is to determine the optimal *amount* of life insurance for an individual. In other words, the problem is to solve

$$\max_{c(t),z} E\Big\{\int_0^{T_x} e^{-\delta t} u(c_t)dt + e^{-\kappa T_x} v(z)\Big\}$$

subject to (i) $E\big(W(T_x)e^{-rT_x}\big) \geq E\big(ze^{-rT_x}\big)$, and (ii) $c_t \geq 0$ for all t and $z \geq 0$.

The Lagrangian for the problem is (ignoring again the non-negativity constraints (ii)),

$$\mathcal{L}(c,z;\lambda) = \int_0^{\tau} u(c_t)e^{-\int_0^t(\delta+\mu_{x+s})ds}dt + v(z)(1-\kappa\bar{a}_x^{(\kappa)}) \\ - \lambda\Big((1-r\bar{a}_x^{(r)})z - \int_0^{\tau}\big(e(t)-c(t)\big)e^{-\int_0^t(r+\mu_{x+s})ds}dt\Big).$$

The first order condition (FOC) in c is the same as for pensions treated above. The FOC in the amount z of life insurance is obtained by ordinary differentiation with respect to the real variable z, which gives

$$v'(z^*) = \lambda \ \frac{1 - r\bar{a}_x^{(r)}}{1 - \kappa\bar{a}_x^{(\kappa)}}.$$

We can thereby determine both the optimal life time consumption, including pension, and the optimal amount of life insurance. An example will illustrate.

Example 3: (The CEIS consumer.) Assume e_t is as in (21), the consumption felicity index is $u(x) = \frac{1}{1-\rho}x^{1-\rho}$, and the life insurance index is $v(x) = \frac{1}{1-\theta}x^{1-\theta}$, where ρ and θ are both time preference parameters. The optimal life insurance amount and optimal consumption/pension are given by

$$z^* = \lambda^{-\frac{1}{\theta}} \Big(\frac{1 - r\bar{a}_x^{(r)}}{1 - \kappa\bar{a}_x^{(\kappa)}}\Big)^{-\frac{1}{\theta}} \quad \text{and} \quad c_t^* = \lambda^{-\frac{1}{\rho}} \ e^{\frac{1}{\rho}(r-\delta)t}. \tag{25}$$

Equality in the 'average' budget constraint (i) determines the Lagrangian multiplier λ. The equation that determines λ is

$$\lambda^{-\frac{1}{\theta}}(1 - r\bar{a}_x^{(r)})\Big(\frac{1 - r\bar{a}_x^{(r)}}{1 - \kappa\bar{a}_x^{(\kappa)}}\Big)^{-\frac{1}{\theta}} + \lambda^{-\frac{1}{\rho}} \ \bar{a}_x^{(r_0)} = y \ \bar{a}_{x:\bar{n}|}^{(r)}. \tag{26}$$

Notice that with life insurance included, the optimal consumption and the pension payments become smaller than without life insurance present, which is seen when comparing the expressions in (25) and (26) with (22). This just tells us the obvious: When some resources are bound to be set aside for the beneficiaries, less can be consumed while alive. The optimal amount in life insurance is an increasing function in income y, and depends on the interest rate r, the pension age n, the time preference parameter ρ as well as the impatience rate δ, the bequest time preference parameter θ and the corresponding impatience rate κ, the insured's age x when initializing the pension and insurance contracts, and the insured's life time distribution through the actuarial formulas in (25) and (26).

Comparative statics in the parameters are not straightforward, and numerical technics may be necessary. As an example, when $\theta = \rho$, it can be seen that the optimal amount of life insurance $z^*(\kappa)$ as a function of the bequest impatience rate κ is increasing for $\kappa \leq \kappa_0$ for some $\kappa_0 > 0$, and decreasing in κ for $\kappa > \kappa_0$. For reasonable values of κ this means that more impatience with respect to life insurance means a higher value z^* of the optimal amount life insurance. $\square$

The above results deviate rather much from the standard actuarial formulas, which is to be expected since the two approaches are different. The actuarial theory is primarily based on the principle of equivalence and time preference neutrality. This is problematic, since time neutral insurance customers would simply not demand any form of pension or life insurance since this individual can handle time substitution very well on on own account. Therefore we assume that the individuals have $\rho > 0$, unlike what is usually assumed in actuarial theory, and use expected utility as our optimization criterion for now.

Without going into details, the effect of a partial increase in the interest rate on the optimal amount of life insurance does not seem to depend on the parameter ψ being larger than, or smaller than one. Rather does the sign of the derivative depend upon the level of the interest rate through some factor $(1 - r\tilde{t})$, where $\tilde{t}$ is some positive break-point-in-time. This derivative tends to be positive when r is small, and negative when r is large. The logic is the following: A future consumption benefit is more valuable today if the interest rate is low than if it is high.

Going back to the actuarial relationship (11), the three quantities p, b and z representing the premium, the pension benefit and the insured amount respectively can in principle be any non-negative numbers satisfying this relationship. In the above example, all these quantities are in addition derived so that expected utility is optimized. The optimal contracts still maintain the actuarial logic represented by the principle of equivalence, which in our case corresponds to the budget constraint (i) on the 'average'. The present analogue to the relationship (11) is:

$$\int_0^n (y - c_t^*) \, \frac{l_{x+t}}{l_x} \, e^{-rt} dt = \int_n^\tau c_t^* \, \frac{l_{x+t}}{l_x} \, e^{-rt} dt + z^*(1 - r\bar{a}_x^{(r)}), \qquad (27)$$

where the constant premium p corresponds to the time varying $p_t = (y - c_t^*)$ for $0 \leq t \leq n$, the constant pension benefit b corresponds to the optimal c_t^* for $n \leq t \leq \tau$, and the number z corresponds to z^* found in (25), where also the optimal pension c_t^* is given. So, even if we use another principle than standard actuarial theory, we agree on the principal structure, represented by the similarity between (27) and (11).

So far the insured amount is still a deterministic quantity, albeit endogenously derived. The reason for the non-randomness in z^* in the present situation is that only biometric risk is considered.

When uncertainty in the financial market is also taken into account, we shall demonstrate that the optimal insured amount becomes state dependent, and the same is true for c_t^*. Both real and nominal amounts are then of interest when comparing the results with insurance theory and practice.

Including risky securities in a financial market is our next topic.

4 The Financial Market

We consider a consumer/insurance customer who has access to a securities market, as well as a credit market and pension and life insurance as considered in the above. The securities market can be described by the vector ν_t of expected returns of N risky securities in excess of the risk-less instantaneous return r_t, and σ_t is an $N \times N$ matrix of diffusion coefficients of the risky asset prices, normalized by the asset process, so that $\sigma_t \sigma_t'$ is the instantaneous covariance matrix for asset returns. Both ν_t and σ_t are assumed to be progressively measurable stochastic processes. Here N is also the dimension of the Brownian motion B.

We assume that the cumulative return process R_t^n is an an ergodic process for each n, where $dX_t^n = X_t^n dR_t^n$ for $n = 1, 2, \ldots, N$, and X_t^n is the cum dividend price process of the nth risky asset.

Underlying is a probability space $(\Omega, \mathcal{F}, P)$ and an increasing information filtration $\mathcal{F}_t$ generated by the d-dimensional Brownian motion, and satisfying the 'usual' conditions. Each price process $X_t^{(n)}$ is a continuous stochastic process, and we suppose that $\sigma^{(0)} = 0$, so that $r_t = \mu_0(t)$ is the risk-free interest rate, also a stochastic process. T is the finite horizon of the economy, so that $\tau < T$. The state price deflator $\pi(t)$ is given by

$$\pi_t = \xi_t e^{-\int_0^t r_s \, ds}, \tag{28}$$

where the 'density' process ξ has the representation

$$\xi_t = \exp\Big(- \int_0^t \eta_s' \cdot dB_s - \frac{1}{2} \int_0^t \eta_s' \cdot \eta_s ds \Big). \tag{29}$$

Here $\eta(t)$ is the market-price-of-risk for the discounted price process $X_t e^{-\int_0^t r_s ds}$, defined by

$$\sigma(\omega, t)\eta(\omega, t) = \nu(\omega, t), \quad (\omega, t) \in \Omega \times [0, T], \tag{30}$$

where the nth component of ν_t equals $(\mu_n(t) - r_t)$, the excess rate of return on security n, $n = 1, 2, \cdots, N$. From Ito's lemma it follows from (29) that

$$d\xi_t = -\xi_t \, \eta_t' \cdot dB_t, \tag{31}$$

i.e., the density ξ_t is a martingale.

The agent is represented by an endowment process e (income) and a utility function $U : L_+ \times L_+ \to R$, where

$$L = \{c : c_t \text{ is progressively measurable, and } E(\int_0^T c_t^2 \, dt) < \infty\}.$$

L_+, the positive cone of L, is the set of consumption rate processes.

The specific form of the function U is as before, namely the time additive and separable one given in (17). Later we change this assumption and consider recursive utility. The remaining life time T_x of the agent is assumed independent of the risky securities X. The information filtration $\mathcal{F}_t$ is enlarged to account for events like $T_x > t$.

This type of situation is called a *temporal* problem of choice. In such a situation is is far from clear that the time additive and separable form of U is the natural representation of preferences (an early reference is here Jan Mossin (1969)).

4.1 The Consumption/Portfolio Choice

The consumer's problem is, for each initial wealth level w, to solve

$$\sup_{(c,\varphi)} U(c) \tag{32}$$

subject to an intertemporal budget constraint

$$dW_t = \big(W_t(\varphi_t' \cdot \nu_t + r_t) - c_t\big)dt + W_t\varphi_t' \cdot \sigma_t dB_t, \qquad W_0 = w. \tag{33}$$

Here $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \cdots, \varphi_t^{(N)})$ are the fractions of total wealth held in the risky securities. The first order condition for the problem (32) is given by the Bellman equation:

$$\sup_{(c,\varphi)} \big\{\mathcal{D}^{(c,\varphi)} J(w,t) - \mu_x(t)J(w,t) + u(c,t)\big\} = 0, \tag{34}$$

with boundary condition

$$EJ(w, T_x) = 0, \quad w > 0. \tag{35}$$

The function $J(w,t)$ is the indirect utility function of the consumer at time t when the wealth $W_t = w$, and represents future expected utility at time t in state w, provided the optimal portfolio choice strategy is being followed from this time on. The differential operator $\mathcal{D}^{(c,\varphi)}$ is given by

$$\begin{aligned}\mathcal{D}^{(c,\varphi)} J(w,t) = {} & J_w(w,t)(w\varphi_t \cdot \nu_t + r_t w - c_t) + J_t(w,t) \\ & + \frac{w^2}{2}\varphi_t' \cdot (\sigma_t \cdot \sigma_t') \cdot \varphi_t \, J_{ww}(w,t).\end{aligned} \tag{36}$$

The problem as it now stands is a non-standard dynamic programing problem, a so called non-autonomous problem. Instead of solving this problem directly, we solve an equivalent one. As is well known (e.g., Cox and Huang (1989) or Pliska (1987)), since the market is complete, the dynamic program (32) - (36) has the same solution as a simpler, yet more general problem, which we now explain.

Also to be noticed at this point is the following: When uncertainty of "gambles" being optimized over resolves at dates in the future, after important decisions must be taken, then use of standard models is suspect and often quite wrong (e.g., Kreps (1988)). "Standard models" here mean the von Neumann-Morgenstern expected utility representation extended to several periods in the additive and time-separable way demonstrated in the above, and also the use of dynamic programming (DP). We know that DP works for the standard model, bur what if the standard model does not work?

4.2 An Alternative Problem Formulation

The problem is here to find

$$\sup_{c \in L} U(c), \tag{37}$$

subject to

$$E\Big\{ \int_0^{T_x} \pi_t c_t \, dt \Big\} \leq E\Big\{ \int_0^{T_x} \pi_t e_t \, dt \Big\} := w \tag{38}$$

Here e is the endowment process of the individual, and it is assumed that e_t is $\mathcal{F}_t$-measurable for all t.

As before, the pension insurance element secures the consumer a consumption stream as long as needed, but only if it is needed. This makes it possible to compound risk-free payments at a higher rate of interest than r_t.

The optimal wealth process W_t associated with a solution c^* to the problem (37)-(38) can be implemented by some adapted and allowed trading strategy φ^*, since the marketed subspace M is assumed equal to L (complete markets). Without mortality this is a well-known result in financial economics.

We claim that by introducing the new random variable T_x this result still holds: In principal mortality corresponds to a new state of the economy, which should normally correspond to its own component in the state price, but the insurer can diversify this type of risk away by pooling over the agents, all in age x, so that the corresponding addition to the Arrow-Debreu state price is only the term $\exp\{-\int_0^t \mu_x(u)du\}$, a non-stochastic quantity. Accordingly, adding the pension insurance contract in an otherwise complete model has no implications for the state price π other than multiplication by this deterministic function, and thus the model is still 'essentially' complete.

4.3 The Optimal Consumption/Pension Problem

The constrained optimization problem (37)-(38) can be solved by Kuhn-Tucker and a variational argument. The Lagrangian of the problem is

$$\mathcal{L}(c;\lambda) = E\Big\{ \int_0^{T_x} \big(u(c_t,t) - \lambda(\pi_t(c_t - e_t))\big)\, dt \Big\}, \tag{39}$$

We assume that the optimal solution c^* to the problem (37)-(38) satisfies $c_t^* > 0$ for a.a. $t \in [0, T_x)$, a.s. Then there exists a Lagrange multiplier, λ, such that c^* maximizes $\mathcal{L}(c;\lambda)$ and complementary slackness holds.

Denoting the directional derivative of $\mathcal{L}(c^*;\lambda)$ in the "direction" $c \in L$ by $\bigtriangledown \mathcal{L}(c^*,\lambda;c)$, the first order condition of this unconstrained problem becomes

$$\bigtriangledown \mathcal{L}(c^*,\lambda;c) = 0 \qquad \text{for all } c \in L \tag{40}$$

This is equivalent to

$$E\Big\{ \int_0^{\tau} \Big(\big(u'(c_t^*)e^{-\delta t} - \lambda \pi_t\big)c(t)\Big) P(T_x > t)dt \Big\} = 0, \qquad \text{for all } c \in L, \tag{41}$$

where the survival probability $P(T_x > t) = \frac{l(x+t)}{l(x)}$. In order for (41) to hold true for all processes $c \in L$, the first order condition is

$$u'(c_t^*) = \lambda e^{\delta t}\pi_t = \lambda e^{-(\int_0^t r_s ds - \delta t)}\xi_t \qquad \text{a.s.,} \qquad t \geq 0 \tag{42}$$

in which case the optimal consumption process is

$$c_t^* = u'^{-1}\Big(\lambda e^{-(\int_0^t r_s ds - \delta t)}\xi_t\Big) \qquad \text{a.s.,} \qquad t \geq 0, \tag{43}$$

where the function $u'^{-1}(\cdot)$ inverts the function $u'(\cdot)$. Comparing the first order condition to the one in (19) where only biometric risk is included, we notice that the difference is the state price density ξ_t in (42). Still mortality does not enter this latter condition.

Differentiation (42) in t along the optimal path c_t^*, by the use of Ito's lemma and diffusion invariance the following stochastic differential equation for c_t^* is obtained

$$dc_t^* = \Big((r_t - \delta)T(c_t^*) + \frac{1}{2}T^3(c_t^*)\,\frac{u'''(c_t^*)}{u'(c_t^*)}\,\eta_t' \cdot \eta_t\Big)\, dt + T(c_t^*)\;\eta_t' \cdot dB_t \tag{44}$$

where $T(c)$ is defined before Example 2. Since there is financial risk present, at first it seems natural to interpret $T(\cdot)$ as the *absolute risk tolerance* function and not as $EIS \cdot c$. A discussion of this issue we return to later.

Comparing with the corresponding differential equation (20) for c_t^* with only biometric risk present, it is seen that including market risk means that the dynamic behavior of the optimal consumption is not so crucially dependent upon whether $r_t < \delta$ at time t or not. This follows since there is an additional term in the drift, and there is a diffusion term present under market risk. Thus the role played by the impatience rate δ is not quite that clear cut with market risk present as it is with only a risk-free credit market.

Notice that when the market-price-of-risk $\eta_t = 0$ of all $t \in [0, T]$ a.s., the two equations coincide.

Below we consider an example in which the felicity index is the same as in Example 3. At this stage the standard model would interpret ρ as the relative risk aversion, call it γ, where $\psi = 1/\gamma$ has the same interpretation as for the deterministic model. Thus time substitution and risk aversion are closely intertwined in the conventional model.

Example 4. (The CRRA/CEIS-consumer.) With the felicity index of Example 3, the optimal consumption takes the form

$$c_t^* = (\lambda e^{-(\int_0^t r_s ds - \delta t)} \xi_t)^{-\frac{1}{\gamma}} \qquad \text{a.s., } t \geq 0. \tag{45}$$

The budget constraint determines the Lagrange multiplier λ, where mortality enters. Suppose we consider an endowment process e_t giving rise to a pension as in (21). Using Fubini's theorem this constraint can be written

$$\int_0^n \Big(y e^{-\int_0^t r_s ds} \frac{l_{x+t}}{l_x} - \lambda^{-\frac{1}{\gamma}} e^{-\frac{\delta t}{\gamma}} E(\pi_t^{(1-\frac{1}{\gamma})}) \frac{l_{x+t}}{l_x} \Big) dt + \int_n^\tau (-1) \lambda^{-\frac{1}{\gamma}} e^{-\frac{\delta t}{\gamma}} E(\pi_t^{(1-\frac{1}{\gamma})}) \frac{l_{x+t}}{l_x}\, dt = 0. \tag{46}$$

For illustration, assume here that the price processes are geometric Brownian motions, the interest rate r is a constant, and the market price of risk η is a constant. By the properties of the state prices π_t and (28) - (31), it then follows that

$$E\big(\pi_t^{(1-\frac{1}{\gamma})}\big) = e^{-[(1-\frac{1}{\gamma})(r + \frac{1}{2}\ \frac{1}{\gamma}\ \eta' \cdot \eta)]t}.$$

Accordingly, the budget constraint can be written

$$y \int_0^n e^{-rt} \frac{l_{x+t}}{l_x}\, dt = \lambda^{-\frac{1}{\gamma}} \int_0^\tau e^{-[\frac{\delta}{\gamma} + (1-\frac{1}{\gamma})(r + \frac{1}{2}\frac{1}{\gamma}\eta' \cdot \eta)]t} \frac{l_{x+t}}{l_x}\, dt.$$

Defining the quantity

$$r_1 = r - \frac{1}{\gamma}(r - \delta) + \frac{1}{2}\frac{1}{\gamma}(1 - \frac{1}{\gamma})\ \eta' \cdot \eta,$$

the Lagrangian multiplier is determined by

$$\lambda^{-\frac{1}{\gamma}} = y\,\frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(r_1)}_x}.$$

From this, the optimal consumption ($t \in [0,n]$) and the optimal pension ($t \in [n,\tau]$) are both given by the expression

$$c^*_t = y\,\frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(r_1)}_x}\; e^{\frac{1}{\gamma}(r-\delta)\,t}\,\xi_t^{-\frac{1}{\gamma}} \quad \text{for all } t \geq 0. \tag{47}$$

which can be compared to (22) giving the corresponding process with only mortality risk present. Notice that this latter formula follows from (47) by setting $\eta = 0$, in which case $\xi_t = 1$ for all t (a.s.) and $r_1 = r_0$.

The expected value of the optimal consumption is given by

$$E(c^*_t) = y\,\frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(r_1)}_x}\exp\Big\{\frac{1}{\gamma}\big(r + \frac{1}{2}\eta'\cdot\eta(1+\frac{1}{\gamma}) - \delta\big)t\Big\}, \tag{48}$$

which is seen to grow with time t (already) when $r > \delta - \frac{1}{2}\eta'\cdot\eta(1+\frac{1}{\gamma})$. When the opposite inequality holds, this expectation decreases with time. In terms of expectations, the crucial border value for the impatience rate δ is no longer r but rather $(r + \frac{1}{2}\eta'\cdot\eta(1+\frac{1}{\gamma}))$ when a stock market is present. $\square$

4.4 Pensions versus ordinary consumption

Now it time we demonstrate why pension insurance exists. This is an extension of the observation made in Section 2.2 about the effect of pooling in a deterministic world.

With pension insurance allowed, the actual consumption at each time t in the life of the consumer is at least as large as the corresponding consumption when the possibility of "gambling" on own life length is not allowed, provided the value of life time consumption w is fixed. This demonstrates a very concrete effect of pooling with market uncertainty allowed.

To this end, consider the random, remaining life time T_x of an x-year old as we have worked with all along, and for comparison, the deterministic life length T, where $T = E(T_x) = \bar{e}_x$ is the expected remaining life time of an x-year old pension insurance customer. For the purpose of this demonstration the above model works just fine.

We consider the situation with a CEIS/CRRA-customer with parameter γ as in Section 4.3, and denote the value of life time consumption w, i.e.,

$$\frac{1}{\pi_0} E\Big(\int_0^{T_x} \pi_t\, c_t^*\, dt \Big) = w.$$

Using (45) this can be written $\lambda^{-\frac{1}{\gamma}}\, \bar{a}_x^{(r_1)} = w$, or

$$\lambda^{\frac{1}{\gamma}} = \frac{\bar{a}_x^{(r_1)}}{w}, \tag{49}$$

where we have set $\pi_0 = 1$ without loss of generality. The corresponding value of life time consumption w for the deterministic time horizon T is determined by

$$\frac{1}{\pi_0} E\Big(\int_0^{T} \pi_t\, c_t\, dt \Big) = w,$$

where it is assumed that in the two situations the budget constraints are the same. Again the optimal consumption/pension c_t is given in (45), however, the Lagrange multipliers determining the optimal consumption/pension are different in the two cases. In order to distinguish, we denote the optimal consumptions by c_t^* and c_t, respectively. The multiplier for the situation with no pension insurance is determined by

$$\lambda_{(T)}^{-\frac{1}{\gamma}} \int_0^{T} e^{-r_1 t}\, dt = w,$$

using Fubini's theorem, which in actuarial notation is equivalent to

$$\lambda_{(T)}^{\frac{1}{\gamma}} = \frac{\bar{a}_{\bar{T}|}^{(r_1)}}{w}. \tag{50}$$

The function $\bar{a}_{\bar{t}|}^{(r_1)} = \int_0^t e^{-r_1 t} dt = \frac{1}{r_1}(1 - e^{-r_1 t})$ is convex in t, which means that $\bar{a}_x^{(r_1)} = E\big(\int_0^{T_x} \pi_t\, c_t^*\, dt \big) = E(\bar{a}_{\bar{T}_x|}^{(r_1)}) < \bar{a}_{\bar{T}|}^{(r_1)}$ by Jensen's inequality, since $T = E(T_x)$. By (49) and (50) this means that $\lambda^{\frac{1}{\gamma}} < \lambda_{(T)}^{\frac{1}{\gamma}}$, and using (45) it follows for all states $\omega \in \Omega$ of the world that

$$c_t^* > c_t \qquad \text{for all } w \text{ and for each } t \geq 0, \tag{51}$$

since the state price density ξ_t is the same in both cases.

With pension insurance available, the individual obtains a higher consumption rate at each time t that he/she is alive. This demonstrates the benefits from pooling when it comes to pensions, and is, presumably, the original reason for its existence.

When the individual with a deterministic horizon dies, the remaining wealth remains with the heirs. This wealth is non-negative by assumption. For the individual with the pension, the remaining wealth at death is distributed among the other pensioners. The individuals in the pool exhaust their life time consumption by gambling on own remaining life time.

5 Implications of the conventional model

Here and in the discussion that follows, we intend to illustrate the issues of intertemporal consumption substitution versus risk aversion. As an alternative derivation of c_t^* in Example 4, the stochastic differential equation (44) for the optimal consumption process is

$$\frac{dc_t^*}{c_t^*} = \big(\frac{r_t - \delta}{\gamma} + \frac{1}{2}\,\frac{1}{\gamma}\,(1 + \frac{1}{\gamma})\,\eta_t' \cdot \eta_t\big)\,dt + \frac{1}{\gamma}\eta_t' \cdot dB_t, \tag{52}$$

The function $T(c) = \frac{c}{\gamma}$ $(= \psi c)$. The "solution" to this stochastic differential equation is

$$c_t^* = c_0 e^{\frac{1}{\gamma}[\int_0^t (r_s - \delta + \frac{1}{2}\eta_s' \cdot \eta_s) ds + \int_0^t \eta_s' \cdot dB_s]}, \qquad t \geq 0.$$

by the Doleans-Dade formula. The initial value c_0 is finally determined by the budget constraint, and (47) would again result in the simple case of constant r and η, and geometric Brownian motion prices, which would then imply that the optimal consumption process is also a geometric Brownian motion as in Example 4.

In society aggregate consumption is observed to be smooth, with a relatively high growth rate, see e.g., Table 1, where the summary statistics of the data used in the Mehra and Prescott (1985)-paper is presented[3]. By $\sigma_{cM}(t)$ we mean the instantaneous covariance rate between the return on the index S&P-500 and the consumption growth rate, in the model a progressively measurable, ergodic process. Similarly, $\sigma_{Mb}(t)$ and $\sigma_{cb}(t)$ are the corresponding covariance rates between the index M and government bills b and between aggregate consumption c and Government bills, respectively. [4]. $\kappa_{M,c}(t)$ is the instantaneous correlation coefficient between the return on the market index and the consumption growth rate.

[3]There are of course newer data sets, and for other countries than the US, but they all retain these basic features. The data is adjusted from discrete-time to continuous-time compounding.

[4]These quantities are "estimated" directly from the original data obtained from Pro-

	Expectation	Standard dev.	covariances
Consumption growth	1.81%	3.55%	$\hat{\sigma}_{Mc} = .002268$
Return S&P-500	6.78%	15.84%	$\hat{\sigma}_{Mb} = .001477$
Government bills	0.80%	5.74%	$\hat{\sigma}_{cb} = -.000149$
Equity premium	5.98%	15.95%	

Table 1: Key US-data for the time period 1889 -1978. Continuous-time compounding. $\hat{\kappa}_{M,c} = .4033$.

In order to match the estimated consumption volatility (3.55 per cent), from (52) we notice that this can be accomplished by a large enough value of the risk aversion γ. This is so since the value of the market-price-of-risk η_t is relatively large, here about .38 and fixed by the summary statistics of Table 1. As it turns out, γ has to be of the order of 26 to match these statistics. (Here, if $d = 1$, we interpret $\sigma_{M,c}(t) = \sigma_M(t)\sigma_c(t)\kappa_{M,c}(t)$.) This leads to a low value for the EIS parameter ψ. In particular this means that $\psi < 1$. If $\psi > 1$, this does not match the low observed consumption volatility of the "representative" consumer.

Many examples have been constructed showing that such a high risk aversion is simply not plausible. Furthermore, an estimate of δ is $\hat{\delta} = -.015$ in order to match the estimated, consumption growth rate of close to two per cent. Normally we think of the impatience rate as a non-negative quantity, since human beings are genuinely impatient.

This calibrated values of the preference parameters constitute a pair of consumption puzzles: The *first* major, empirical problem with the conventional model is to explain the smooth path of the aggregate consumption growth rate in society. From (52), where $\sigma_c(t) = \eta_t/\gamma$, for the estimated value of η_t, this requires a large value of γ to match the low estimate of the consumption volatility.

The *second* major problem with the conventional model is to explain the relatively large estimate of the growth rate of aggregate consumption in society for plausible values of the parameters. For the estimated value of η_t, and the large value of γ required to match the low estimated volatility, this requires a very low value of the impatience rate δ in (52), in fact it has to be negative.

Solving the first problem also solves the second, so there is really one major puzzle.

The conventional model predicts too large consumption growth volatility,

fessor Rajnish Mehra, using the ergodic assumption, and estimates are denoted by $\hat{\sigma}_{M,c}$, etc.

and too low consumption growth rate for more reasonable values of the preference parameters. Including pension insurance for the representative agent does not change this. Consumption substitution is carried out by the contract, and life time consumption is smoothed, but the consumption growth rate is not. We conclude that the model can not explain well the observed data.

These two consumption puzzles are of course related to the celebrated "Equity Premium Puzzle" and the "Risk Free Rate Puzzle", see Mehra and Prescott (1985) and Weil (1989)). With equilibrium imposed, these two sets of puzzles are in fact identical.

Both the insurance industry and public institutions contribute to completion of the real world markets, and to consumption substitution during the life cycles of the citizens. The conventional model can not explain the observed level of smoothing in the consumption growth rate. A major weakness with this model is that two different abilities of human beings are too tightly linked together; risk aversion equals time preference, i.e., $\rho = \gamma$, or $\psi = 1/\gamma$. In particular, these two properties of an individual should be separated.

The above leads us to consider alternative types of the representation of preferences. The one we find of particular interest in pension insurance is recursive utility, which allows us to separate consumption substitution from risk aversion. The resulting model gives an optimal consumption that involves more smoothing than the present model. As a consequence it fits much better the summary statistics of Table 1 for reasonable parameter values. This is the topic of our companion paper.

For an insurance company the implications of the observations from the present model may be several: In real life companies meet many different types of customers, demanding different pension insurance contracts. The above individual is rather extreme, and can not really be taken seriously as the "representative customer". If so, a customer like this individual would need assistance both in substituting consumption across time, and also in saving/investment decisions. A defined benefit pension plan would clearly be appropriate. However, the insurance industry can not tailor make their contracts to an "individual" who constitutes an empirical puzzle.

6 Business cycles included

In order to demonstrate how robust the puzzle of the previous section is, let us assume that the stock market index is a mean reverting process, and see if this changes anything. Since business cycles exist in the real world, this will make our model of the market index more realistic. Business cycles should somehow be reflected in the data, which a realistic model should also account for. In Example 4 we made the assumption that the market index is a geometric Brownian motion.

As a concrete illustration, imagine that the stock market index satisfies the following dynamic equation

$$dX_t = \kappa_t(\alpha_t - \ln X_t)X_t dt + \sigma_M(t)X_t dB_t \tag{53}$$

Here κ_t and α_t are two deterministic processes, that could be just constants, and $\sigma_M(t)$ is the volatility of the return rate on the market index, satisfying usual conditions. The price process X_t is a strictly positive process such that $Y_t = \ln X_t$ is an Ornstein-Uhlenbeck (OU)-process. The mean reversion effect in this model is higher for large values of X_t than for low. For α, κ and σ_M constants, the solution of (53) is

$$X_t = X_0 \exp\Big\{\big(\alpha - \frac{1}{2}\frac{\sigma_M^2}{\kappa}\big)\big(1 - e^{-\kappa t}\big) + e^{-\kappa t}\sigma_M \int_0^t e^{\kappa s} dB_s\Big\}, \tag{54}$$

in which case

$$dY_t = \kappa(\alpha - Y_t)dt + \sigma_M dB_t, \tag{55}$$

which is the dynamics of an OU-process. In this model the market-price-of-risk is $\eta_t = (\kappa(\alpha - \ln X_t) - r_t)/\sigma_M(t)$. Proceeding as in Section 4.3 we now use the FOC (42) i.e., $c_t^* = (\lambda e^{\delta t}\pi_t)^{-\frac{1}{\gamma}} := f(\pi, t)$, where

$$d\pi_t = -\pi_t(r_t dt + \eta_t dB_t).$$

By Ito's lemma, since

$$f_\pi(\pi, t) = -\frac{1}{\gamma}(c_t^*)^{(1+\gamma)}\lambda e^{\delta t}, \quad f_{\pi,\pi}(\pi, t) = \frac{1}{\gamma}(\frac{1}{\gamma} + 1)(c_t^*)^{(1+2\gamma)}\lambda^2 e^{2\delta t} \quad \text{and}$$

$$f_t(\pi, t) = -\frac{\delta}{\gamma}c_t^*,$$

this gives

$$\frac{dc_t^*}{c_t^*} = \big(\frac{r_t - \delta}{\gamma} + \frac{1}{2}\,\frac{1}{\gamma}\,(1 + \frac{1}{\gamma})\,\eta_t' \cdot \eta_t\big)\,dt + \frac{1}{\gamma}\eta_t' \cdot dB_t. \tag{56}$$

This is equation (44) with the present market-price-of-risk η and the present risk tolerance function T. As can be seen, this is also the same equation for the optimal consumption c_t^* as (52), again with the present process η_t. Since

$$\eta_t = \frac{\kappa_t(\alpha_t - \ln X_t) - r_t}{\sigma_M(t)} = \frac{\mu_M(t) - r_t}{\sigma_M(t)}, \tag{57}$$

where $\mu_M(t) = \kappa_t(\alpha_t - \ln X_t)$ is the return rate of the market index, the consumption puzzle with mean reversion is seen to be the same as the corresponding puzzle with geometric Brownian motion in Example 4.

Since no distributional assumptions were made in equation (44), we could equally well have taken this equation as a starting point for our present investigation. In other words, the consumption puzzle remains for the present set of preferences regardless of the form of the drift and diffusion terms of X so long as these satisfy standard conditions for existence of solutions of the corresponding stochastic differential equations.

Another question is the impact of business cycles on welfare. This has been the topic of much research over the last 40 years. This cycle around the secular trend has negative impact on consumer welfare. Suppressing it, i.e., smoothing out the business cycles, would be beneficial to consumers who dislike consumption fluctuations around the optimal growth rate $\mu_c(t)$ [5]. As mentioned, the insurance industry can contribute by making reserves in good times, i.e., by time diversification.

A last point: From analyses in the frequency domain (e.g., Dew-Becker and Giglio (2013)), we know that for the standard power utility the only thing that determines the price of risk for a shock is how it affects consumption today, while under standard recursive utility long-run risks matter. Recall that the negative shock of the finance crisis in 2007 had a longer lasting effect on consumption than just one year, which is more in line with the latter preference specification. However, much remains to be done in this direction.

7 Discussion of the optimal pension

Abstracting from the consumption puzzle on the aggregate level, the model is still considered to give interesting results on an individual level for many other issues.

We choose to refer to risk preference instead of time preference, but as we saw previously, we must be keenly conscientious about the difference. Observe that when stock market uncertainty is present, since $\gamma > 0$, the solution in (45) tells us that when state prices π_t are low, optimal consumer is high, and vice versa. State prices reflect what the representative consumer is willing to pay for an extra unit of consumption; in particular it is convenient to think of π_t high in "times of crises" and low in "good times".

[5] This property of the consumer has to do with *time preference*, defined as ψ^{-1}. Ideally this should be different from risk aversion γ.

In real terms this property of pensions is the same as for optimal consumption. In times of crises the pensions are lower than in good times. This merely explains the intrinsic logic of this treatment, namely that society can only pay the pensioners what the economy can manage at each time. To the extent that this also happens in real life, this is partly a consequence of the way insurance companies and governments manage resources by yearly budgets.

Insurance companies, for example, pay the pensions from funds, which in bad times are lower than in good times. Such companies have the possibility and ability, however, to take a long term view and build reserves in good times, thereby smoothing premium reserves across time.

With this perspective in mind, insurance companies could consider providing the type of pension and life insurance contracts that many people seem to prefer, namely that of smoothing life time consumption across both time, and states of nature. Since individuals have a shorter time perspective than the insurance industry, individuals can not "time diversify" the way the industry can. Again this is an argument for pooling, but more that that: An insurance company can interchange time integrals with state integrals under ergodicity (The Gibbs Conjecture) better than an individual.

Ordinary state pensions are paid out each year to the whole generation of pensioners. If this is done on a year by year basis with yearly budget constraints, this will naturally lead to real fluctuations in benefits. If aggregate consumption in society is down in one particular year, everyone is in principle worse off, simply by the *mutuality principle*. This appears very different if also governments chose to take a long term view and smooth across time as well as over the states of nature, something a government should be able to accomplish, and many nations actually do this to a certain degree. A deeper discussion of this topic would lead us into business cycles, fiscal policy and macroeconomics, which is beyond the scope of this presentation (see e.g., Rodden et. al. (2003)).

7.1 Comparative statics

In Section 3 we considered what happens to consumption, in a pure credit market with only mortality risk when, for example, the interest rate increases. In the present we have the possibility to investigate what happens to consumption when conditions in the market for risky assets change. Such partial analyses may have limited validity, since the actual capital market should be in some sort of equilibrium. In such an environment an increase in, for example, the market-price-of-risk may stem from increased uncertainty in relation to the aggregate consumption in society, which in its turn will lower the equilibrium interest rate.

At the risk of violating such principles, let us nevertheless consider the *partial effect* on the expected value of the optimal consumption at time t, as seen from time $t = 0$, of an increase in the market-price-of-risk. For simplicity of exposition we assume there to be only one risky asset ($N = 1$), where σ_t, ν_t and r_t are all deterministic and constant in time. We then get

$$\frac{\partial E(c_t^*)}{\partial \eta} = y\,\frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(r_1)}_{x}} \exp\Big\{\frac{1}{\gamma}\big(r + \frac{1}{2}\eta' \cdot \eta(1+\frac{1}{\gamma}) - \delta\big)t\Big\}\,\eta\,\frac{1}{\gamma}(1+\frac{1}{\gamma})(t - \tilde{t_1}), \quad (58)$$

where

$$\tilde{t_1} = \frac{\frac{1}{\gamma} - 1}{\frac{1}{\gamma} + 1}\,\tilde{t_2}$$

and $\tilde{t_2}$ is determined by the equality

$$\int_0^\tau s\,\frac{l_{x+s}}{l_x}\,e^{-r_1 s}\,ds = \tilde{t_2}\int_0^\tau \frac{l_{x+s}}{l_x}\,e^{-r_1 s}\,ds$$

by the first mean value theorem for integrals. We notice that only when the parameter $\gamma \leq 1$, or $\psi \geq 1$, is the break-point-in-time $\tilde{t_1} \geq 0$, in which case the expected consumption decreases for $t \leq \tilde{t_1}$ and increases for $t > \tilde{t_1}$ with an increase in η, ceteris paribus. This should be compared to the substitution effect in Example 2, and must be attributed to the EIS-interpretation of $\psi = 1/\gamma$. When $\psi < 1$ on the other hand, an increase in the market-price-of-risk η leads to an increase in the expected consumption for all $t > 0$, and there is no transparent substitution between consumption early and late in life. The income effect then dominates.

An increase in η could also mean a decrease in the volatility σ in the stock market (recall that $\sigma_M \eta = (\mu_M - r)$), in which case more is invested in the stock market relative to the bond market. The individual with $\gamma < 1$ would then invest, consume less earlier and more later. The risk averse individual with $\gamma > 1$ would miss this opportunity, according to the standard model.

7.2 Pensions in nominal terms

Pensions (and insurance payments) are usually not made in real, but in nominal terms. There exist index-linked contracts, but these are still more the exception than the rule. In nominal terms the optimal consumption is given by $c_t^* \pi_t$.

For the model of Example 4, the nominal pension is

$$c_t^* \pi_t = (\lambda e^{\delta t})^{-\frac{1}{\gamma}}\;\pi_t^{(1-\frac{1}{\gamma})}.$$

Here the value $\gamma = \psi = 1$ is again seen to be a border value of these two parameters in the sense that for $\gamma > 1$ ($\psi < 1$) both optimal consumption and pensions in nominal terms are countercyclical. This can give rise to an *illusion* of being insured against times of crises.

People with $\gamma < 1$ ($\psi > 1$) experience no such illusion, since nominal amounts behave as real amounts with respect to cycles in the economy. In the situation when $0 < \gamma < 1$ the agent is sometimes called *risk tolerant.* Notice that when $\gamma = 1$ the nominal consumption does not vary with the state price π_t and is in addition deterministic.

The optimal pension problem has no solution unless the agent is risk averse, i.e., $\gamma > 0$.

Independent studies indicate a range of γ from zero to ten, where a value between one half and three is considered both moderate and plausible[6].

8 The connection to actuarial theory and insurance practice

In standard actuarial theory the nominal pension is nonrandom, at least this is what most textbooks on the subject take as a premise. Referring to the above standard theory, this is only consistent with $\gamma = 1$, corresponding to logarithmic utility, the case when the substitution effect and the income effect cancel in the standard model. In addition this theory commonly uses the principle of equivalence to price insurance contracts, where the state price density implicitly is set equal to a constant, i.e., $\xi_t \equiv 1$. This implies that the agent is really risk neutral, so $\gamma = 0$ follows, and the conventional model breaks dow. Thus there seems to be an inconsistency between the standard life cycle model and actuarial text-book theory.

In insurance practice, which actuaries are primarily engaged in, let us again distinguish between the two main types of contracts; (a) defined benefits, and (b) defined contributions. With regard to the first, before possible profit sharing the nominal value is usually taken to be constant in the insurance contracts, although as we have noticed, sometimes contracts are offered where the real value is approximately constant. A deterministic contract is not consistent with any finite value of γ for the standard model.

Attached to this contract is usually a return rate guarantee. Many life insurance companies are having difficulties with this guarantee in times when

[6]It may be of interest to notice that Kimball et.al. (2008) indicate a value of the relative risk aversion between 3 and 8, based on responses to hypothetical income gambles in the Health and Retirement Study, a large-scale US-survey.

the stock market is down. Lately, in times of crises, this tends to go together with a low interest rate (like in the financial crisis of (2007, -)) due to government interference. In such cases life insurance companies suffer twofold, and must rely on built-up reserves before, possibly, equity is being used. These problems seem closely connected to a regulatory regime with focus on yearly performance, for contracts that are intrinsically long term.

Defined contribution contracts are actively marketed by the insurance companies at the present. For such contracts the insurance customers take all the financial risk, and mainly mortality risk remains with the companies. There is no rate of return guarantees, and the contract functions much like unit linked pension contract, or simply like a mutual fund. Thus the nominal, as well as the real pensions are state dependent, in accordance with the basic theory outlined above. In neither case does a guaranteed return enter the optimal pension contract. A guarantee affects the insurance company's optimal portfolio choice plan. Typically, due to the nature of the guarantees and regulatory constraints, the companies are led to sell when the market goes down, and buy when the market rises. With a constant market volatility, this is just the opposite of what is known to be optimal, to be discussed later. However, typically market volatility increases in crises, in which case it may be optimal to sell.

Guarantees seem attractive to customers for a variety of reasons, so such contracts are not likely to disappear from the market: By the principle of *dynamic consistency*, if there is some product that enough people want, eventually there will be a market for this product. Insurers are reluctant to offer such contracts at the present, but this may well change in the future.

There are different reasons why guarantees originated in the life and pension insurance business. In Norway, as the story goes, it became part of the legal terms of the contracts, more or less by an oversight, in times where the short term interest rate was considerably higher that the 4% that was generally employed in the premium calculations on which the standard actuarial tables were based. Also demand from customers likely played a role.

During the financial crisis of 2007 and onwards, casual observations seem to suggest that many *individuals* would rather prefer the defined benefit type to the other. As an example[7], the employees of a life and pension insurance company would rather prefer a collective defined benefit pension plan, but were voted down by the board. Collective pension plans organized by firms on behalf of their workers, are almost exclusively defined contribution plans these days (at least in Norway), which appear to be the least costly of the two for the firms, and also the preferred choice to offer by the insurance

[7] a case known to the author

companies at the present.

In times of crises, defined benefit pension contracts seem most attractive to the customers, at least as long as they ignore the possibility that their insurance company may go bankrupt. In the crisis referred to above, some life insurance companies failed. However, a great number of individuals throughout the world lost parts of, or even their entire pensions due to the fall in the stock market, for holders of defined contribution pension plans. In times of rising stock prices, on the other hand, such contracts may seem attractive to many individuals. What alternative the individuals find best may thus seem to depend upon where in the business cycle an individual happens to retire. In isolation, this does not seem like a sound principle.

In practice, when a pension insurance customer approaches retirement age, the financial risk of the individual's premium reserve is gradually decreased by the company to avoid these kind of problems.

In the next section we present a simple one-period model that points in the same direction.

9 A simple one-period model. The "timeless" case.

In real life consumers are likely to separate consumption decisions from pensions. A reasonable pension may then, at least partly, be regarded as an insurance against a bad state in the economy when the consumer retires.

In this regard it may be useful to return to the standard actuarial model of Section 2.1, where equation (8) prescribes a fixed yearly pension b, when optimal consumption is taken as given. In the present setting where market risk is included, in order to obtain a insurance effect of a pension we should, perhaps, bring in insurance companies explicitly in the model and consider Pareto optimal risk sharing, but that is beyond the scope of the present presentation.

What we choose to focus on is the following. Consider the insurance problem in the simple setting of a one-period model (a timeless problem). Let W be the state dependent wealth at time one (which is also the consumption then) without pension insurance. Suppose the pension insurance buyer can purchase a pension contract specifying the amount Y to be paid out at time one at a premium p paid at time zero. Here the pension amount $Y = Y(\omega)$ is also state dependent seen from time zero. The state price in the economy is denoted by π, a random variable, and here taken as exogeneous. The optimal pension amount Y is then a solution of the following problem,

$$\max_Y Eu(W+Y-p) \quad \text{subject to} \quad E\big(\pi \cdot (W+Y-p)\big) \leq E\big(\pi \cdot W\big). \tag{59}$$

Here the inequality is the agent's budget constraint and the utility function u satisfies $u' > 0, u'' < 0$. The Lagrangian of the problem (59) is

$$\mathcal{L}(Y;\lambda) = E\big(u(W+Y-p) - \lambda\pi \cdot (Y-p)\big).$$

The directional derivative of the Lagrangian at the optimal Y, denoted by Y^*, in the direction Z is

$$\triangledown\mathcal{L}(Y^*;\lambda;Z) = E\Big(\big(\frac{\partial u}{\partial Y}(W+Y^*-p) - \lambda\pi\big)Z\Big).$$

The first order condition is then

$$\triangledown\mathcal{L}(Y^*;\lambda;Z) = 0 \quad \text{in all 'directions' } Z \in L^2,$$

which implies that

$$u'(Y^* + W - p) = \lambda\pi$$

or

$$Y^* = (u')^{-1}(\pi\lambda) - (W-p) \quad a.s. \tag{60}$$

This equation tells us that the optimal pension amount is negatively correlated with random endowment W. As we have pointed out, this is a desirable property of a pension insurance. Since the budget constraint is obtained with equality ($u' > 0$), the premium $p = E(\pi Y^*)$.

Normally the consumption endowment W results from savings in the security market, and if this is done in an 'optimal' way W is negatively correlated with the state price π as earlier explained. Under risk aversion, the inverse function of u' is a decreasing function. This means that when the state price π increases, and consequently $(W-p)$ decreases, on the average, the term $(u')^{-1}(\pi\lambda)$ decreases, and the term $-(W-p)$ increases, on the average. As a consequence the positive correlation between Y^* and the security market is reduced, which is what risk averse pension customers presumably want. Notice that for timeless problems the Eu-theory seems to work just fine.

10 Including Life Insurance

10.1 The conventional model

We are now in position to analyze life insurance in the setting of the life cycle model. The contracts we derive here are idealizations. The results may give useful information to the life insurance industry about what contracts to offer.

We assume that the felicity index u and the utility function v are as in Example 3 of Section 3.1. The problem can then be formulated as follows:

$$\max_{z,c\geq 0} E\Big\{ \int_0^{T_x} e^{-\delta t}\frac{1}{1-\gamma}c_t^{1-\gamma}dt + e^{-\kappa T_x}\frac{1}{1-\theta}z^{1-\theta}\Big\}$$

subject to

$$E\Big\{e^{-rT_x}W(T_x)\Big\} \geq E\Big\{\pi_{T_x}z\Big\},$$

where z is the amount of life insurance, here a random decision variable. The Lagrangian of the problem is:

$$\mathcal{L}(c,z;\lambda) = E\Big\{ \int_0^{\tau} e^{-\delta t}\frac{1}{1-\gamma}c_t^{1-\gamma}\frac{l_{x+t}}{l_x}dt + e^{-\kappa T_x}\frac{1}{1-\theta}z^{1-\theta} - \lambda\big[\pi_{T_x}z - \int_0^{\tau}(e_t - c_t)\frac{l_{x+t}}{l_x}dt\big]\Big\}.$$

The first order condition in c is:

$$\bigtriangledown_c\mathcal{L}(c^*,z^*;\lambda;c) = 0, \quad \forall c\in L_+$$

which is equivalent to

$$E\Big\{ \int_0^{\tau}\big((c_t^*)^{-\gamma}e^{-\delta t} - \lambda\pi_t\big)c_t\frac{l_{x+t}}{l_x}dt\Big\} = 0, \quad \forall c\in L_+$$

and this leads to the optimal consumption/pension

$$c_t^* = \big(\lambda e^{\delta t}\pi_t\big)^{-\frac{1}{\gamma}} \quad \text{a.s. } t\geq 0$$

as we have seen before in (45). The first order condition in the amount of life insurance z is:

$$\bigtriangledown_z\mathcal{L}(c^*,z^*;\lambda;z) = 0, \quad \forall z\in L_+$$

which is equivalent to

$$E\Big\{\big((z^*)^{-\theta}e^{-\kappa T_x} - \lambda\pi_{T_x}\big)z\Big\} = 0, \quad \forall z\in L_+ \tag{61}$$

Notice that both z^* and z are $\mathcal{F}\vee\sigma(T_x)$ - measurable. For (61) to hold true, it must be the case that

$$z^* = \big(\lambda e^{\kappa T_x}\pi(T_x)\big)^{-\frac{1}{\theta}} \quad \text{a.s.,} \tag{62}$$

showing that the optimal amount of life insurance z^* is a state dependent $\mathcal{F}_{T_x}$ - measurable quantity.

One may wonder which of time preference or risk preference is the correct interpretation of the parameters θ and γ. (This will be clear once we turn to recursive utility.)

If the state is relatively good at the time of death, the state price π_{T_x} is then low and $(\pi_{T_x})^{-\frac{1}{\theta}}$ is relatively high (when $\theta > 0$). Thus this life insurance contract covaries positively with the business cycle. In practice this could be implemented by linking the payment z^* to an equity index.

One can again wonder how desirable this positive correlation with the economy is. For optimal consumption we found it quite natural, but not so for pensions. Similarly, life insurance possess the characteristics of an ordinary, (non-life) insurance contact. In some cases it may seem reasonable that a life insurance contract is countercyclical to the economy, thereby providing real insurance in time of need. For this to be the result, however, the function v must be convex, corresponding to *risk proclivity* which here means that $\theta < 0$, but risk loving people do not buy insurance.

The expected value of z^* is found by conditioning, assuming that ν, η and r are all deterministic constants. It is given by the formula

$$E(z^*) = \lambda^{-\frac{1}{\theta}} \int_0^{\tau} \exp\Big\{\frac{1}{\theta}\big(r + \frac{1}{2}\eta' \cdot \eta(1 + \frac{1}{\theta}) - \kappa\big)t\Big\} \frac{l_{x+t}}{l_x} dt. \tag{63}$$

For a given value of budget constraint (λ), this expectation is seen to be larger if $r + \frac{1}{2}\eta' \cdot \eta(1 + \frac{1}{\theta}) > \kappa$ than if the opposite inequality holds. As for pensions, in terms of expectation has the impatience cutt-off-point increased from r to $(r + \frac{1}{2}\eta' \cdot \eta(1 + \frac{1}{\theta}))$. In other words, not only the market interest rate r, but also the market-price-of-risk and the relative risk aversion of the function v determines what it means to be impatient, when a stock market is present.

Using the budget constraint with equality, we find an equation for the Lagrange multiplier λ;

$$E\Big\{\pi_{T_x} z^* - \int_0^{\tau} (e_t - c_t^*)\pi_t \frac{l_{x+t}}{l_x} dt\Big\} = 0.$$

With a constant income of y up to the time n of retirement, and an optimal pension c_t^* thereafter as in (21), we obtain the equation

$$\lambda^{-\frac{1}{\theta}}(1 - r_2 \bar{a}_x^{(r_2)}) + \lambda^{-\frac{1}{\gamma}} \bar{a}_x^{(r_1)} = y \bar{a}_{x:\bar{n}|}^{(r)},$$

where

$$r_1 = r - \frac{1}{\gamma}(r - \delta) + \frac{1}{2}\eta' \cdot \eta \, \frac{1}{\gamma} \, (1 - \frac{1}{\gamma})$$

as in Section 4.3, and

$$r_2 = r - \frac{1}{\theta}(r-\kappa) + \frac{1}{2}\eta' \cdot \eta \, \frac{1}{\theta}\,(1-\frac{1}{\theta}).$$

In the special situation where $\kappa = \delta$ and $\theta = \gamma$ so that $u = v$, it follows that $r_1 = r_2$ and

$$\lambda^{-\frac{1}{\gamma}} = \frac{y\bar{a}^{(r)}_{x:\bar{n}|}}{(1+(1-r_1)\bar{a}^{(r_1)}_x)}.$$

It is at this point that pooling takes place in the contract. In this situation the optimal consumption/pension is given by

$$c^*_t = \frac{y\bar{a}^{(r)}_{x:\bar{n}|}}{(1+(1-r_1)\bar{a}^{(r_1)}_x)}\, e^{((r-\delta)/\gamma)t}\, \xi_t^{-\frac{1}{\gamma}}, \tag{64}$$

and the optimal amount of life insurance at time T_x of death of the insured is

$$z^* = \frac{y\bar{a}^{(r)}_{x:\bar{n}|}}{(1+(1-r_1)\bar{a}^{(r_1)}_x)}\, e^{((r-\delta)/\gamma)T_x}\, \xi_{T_x}^{-\frac{1}{\gamma}}. \tag{65}$$

This means that the optimal amount of life insurance is determined jointly, through the constant λ, with the optimal consumption/pension.

We claim that these contracts represent an innovation in the theory of ordinary life insurance.

In both (64) and (65) the risk aversion parameter γ appears. In the recursive model, we shall se that this interpretation is only reasonable in the last formula regarding life insurance. Thus pensions have to do with consumption substitution, life insurance with risk attitudes.

10.2 Discussion of state dependent life insurance contracts

If large parts of the population buy life insurance products, a positive correlation with the business cycle is a natural property of the life cycle model. If we introduce insurance companies, this corresponds to no building of reserves. Unlike pension insurance, however, life insurance is a product that not everybody seems to demand. We can single out two different family situations where life insurance is of particular interest. The first concerns a relatively young family with small children. Then one of the parents can usually not work full time, which means that the other is the main provider. If this person dies, in for example an accident, this is of course dramatic for this family. Life insurance then plays the role of insurance for the loss of the remaining life time income. As can be seen from the expression in (65), the insured amount is proportional to the present value at time zero of life time income $y\bar{a}^{(r)}_{x:\bar{n}|}$. If death comes early, T_x is relatively small so the factor $e^{((r-\delta)/\gamma)T_x}$ is close to one.

The other situation we have in mind is the traditional one with a bequest motive, usually meaning that an older person wants to transfer money to his or her heirs. The social need for this insurance may seem less obvious than in the first situation described. Here the factor $e^{((r-\delta)/\gamma)T_x}$ may be large for the patient life insurance customer, implying a large insured sum to the beneficiaries. Despite of all the good reasons for a life insurance contract for the young family, its seems far less widespread than life insurance with the bequest motive, which is somewhat ironic.

In climate economics the bequest idea could be interesting in the following sense. By paying a premium today (e.g., by reducing consumption and utility now), one may "roll over" a more sustainable society to future generations by "inter-personal transfers".

One objection to the optimal solutions (62) and (65) is that the amount payable has not been subject to "enough pooling" over the individuals. The pooling element is present, since it is used in the budget constraints, but the amount payable is here crucially dependent on the actual time of death T_x of the insured, which is unusual in both life insurance theory and practice.

Focusing on the standard model, one alternative approach is to integrate out mortality in the first order condition (61). Notice that this is strictly speaking not the correct solution to the optimization problem, but must instead be considered as a suboptimal pooling approximation. This results in the following approximative first order condition:

$$E_{z,z^*}\Big\{\Big((z^*)^{-\gamma}(1-\delta\bar{a}^{(\delta)}_x) - \lambda\int_0^\tau \pi_t f_x(t)dt\Big)z\Big\} = 0, \quad \forall z,$$

assuming again that $\kappa = \delta$ and $\theta = \gamma$. The solution to this problem, also a random variable, is given by

$$\bar{z}^* = \Big(\frac{\lambda\int_0^\tau \xi_t e^{-rt} f_x(t)dt}{1-\delta\bar{a}^{(\delta)}_x}\Big)^{-\frac{1}{\gamma}} \qquad \text{a.s.} \tag{66}$$

However, this contract is seen to depend on the state of the economy from time 0 when the insured is in age x, to the end of the insured's horizon τ. At time of death T_x $(<\tau)$ this quantity is not entirely known, which is a consequence of our approximative procedure. Ignoring this information problem for the moment, by employing the budget constraint, the Lagrange multiplier λ is found as

$$\lambda^{-\frac{1}{\gamma}} = \frac{y\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(r_1)}_x + \frac{E[(\int_0^\tau \xi_t e^{-rt} f_x(t)dt)^{(1-\frac{1}{\gamma})}]}{(1-\delta\bar{a}^{(\delta)}_x)^{-\frac{1}{\gamma}}}} \tag{67}$$

Inserting λ from (67) into (66), the suboptimal insured amount results.

When stock market uncertainty goes to zero, i.e., when $\xi_t \to 1$ a.s., $\bar{z}^*$ converges to the corresponding contract of Section 3.1 when only biometric risk is present.

We can derive an insured amount z^{**} that is consistent with the information available at time of death of the insured as the following conditional expectation

$$z^{**} := E\{\bar{z}^* | \mathcal{F}_{T_x}\}.$$

This is a random variable at the time when the life insurance contract is initialized, and an observable quantity at the time of death of the insured, and thus solves the information problem.

Note that this contract would benefit the young family in the case of early death of the provider, since those who die early are subsidized by those who live long when the insured sum is subject to enough averaging.

The advantage with this contract is that it takes into account pooling over life contingencies at two stages of the analysis. Furthermore it is consistent with the standard analysis when there is "no market risk in the limit".

11 The optimal portfolio choice problem

We have barely touched upon the portfolio choice problem in Section 4.1, but could there proceed without really having to solve it. This is due to the fact that in the model we discuss, we may separate the the consumer's portfolio choice problem from his or her optimal consumption choice. In the present section we do solve the investment problem explicitly. For this we need the agent's net wealth W_t at time t. For the CRRA-consumer of the standard model, it is given by

$$W_t = \frac{1}{\pi_t} E_t \Big\{ \int_t^{T_x} \pi_s c_s^* ds \Big\} = \frac{1}{\pi_t} E_t \Big\{ \int_t^{T_x} \pi_s^{(1-\frac{1}{\gamma})} \lambda^{-\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}s} ds \Big\},$$

where we have used (45). Here E_t means conditional expectation given the information filtration $\mathcal{F}_t \vee (T_x > t)$, i.e., given the financial information available at time t and the fact that the individual is alive then. Recalling that at time t the agent is in age $x+t$, we get, using Fubini's theorem

$$W_t = \frac{1}{\pi_t}\lambda^{-\frac{1}{\gamma}} \int_t^{\tau} E_t(\pi_s^{(1-\frac{1}{\gamma})})\, e^{-\frac{\delta}{\gamma}s}\, \frac{l_{x+s}}{l_{x+t}}\, ds.$$

Provided $\sigma_t = \sigma$, $\nu_t = \nu$ and $r_t = r$ are all deterministic constants, the conditional expectation appearing in the integrand is computed as follows:

$$E_t(\pi_s^{(1-\frac{1}{\gamma})}) = E_t\big(\pi_t^{(1-\frac{1}{\gamma})} e^{(1-\frac{1}{\gamma})(-r-\frac{1}{2}\eta'\cdot\eta)(s-t)+(1-\frac{1}{\gamma})\eta'\cdot(B_s-B_t)}\big) =$$

$$\pi_t^{(1-\frac{1}{\gamma})} e^{-[(1-\frac{1}{\gamma})r+\frac{1}{2}\frac{1}{\gamma}(1-\frac{1}{\gamma})\eta'\cdot\eta](s-t)},$$

where we have used the lognormal representation for the state price π and the moment generating function of the normal distribution. This gives for the wealth process

$$W_t = \pi_t^{-\frac{1}{\gamma}} \lambda^{-\frac{1}{\gamma}}\, e^{-\frac{\delta}{\gamma}t}\, \bar{a}_{x+t}^{(r_1)} = c_t^*\, \bar{a}_{x+t}^{(r_1)}, \tag{68}$$

where r_1 is as given in Section 4.3. This shows that the wealth at any time t in the life of the consumer, who is then in age $(x+t)$, is equal to the actuarial value of receiving the optimal consumption c_t^* per time unit for the rest of his or her life, discounted at the rate r_1. For logarithmic utility, $r_1 = \delta$ the subjective interest rate; when $\gamma \neq 1$ this discount rate depends on the volatility of the state prices, or the market price of risk η, δ, r as well as of γ. In fact, r_1 can be interpreted as the a risk adjusted return rate.

Using the dynamics for c_t^* given in (47), by Ito's lemma we obtain the following dynamic representation for the wealth W_t:

$$dW_t = \mu_W(t)dt + \frac{1}{\gamma}W_t\eta\cdot dB_t,$$

for some drift term $\mu_W(t)$. Comparing this to the intertemporal budget constraint (33) of Section 4.1, we may apply diffusion invariance to determine the the optimal fractions $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \cdots, \varphi_t^{(N)})$ of total wealth held in the risky securities at each time t. By equating the two diffusion terms, we obtain that

$$\frac{1}{\gamma}\eta = \varphi_t\cdot\sigma.$$

and recalling that $\sigma\eta = \nu$, it follows from this that the optimal investment fractions are

$$\varphi = \frac{1}{\gamma}(\sigma\sigma')^{-1}\nu, \tag{69}$$

where ν, with components $\nu_n = \mu_n - r$, $n = 1, 2, \cdots N$, is the vector of risk premiums for the N risky securities. These ratios are all seen to be constants, meaning that they do not depend upon the age $(x + t)$ of the investor, the state of the economy π, or on the investor's death intensity μ_{x+t}.

This result is the same as the one found by Mossin (1968), Samuelson (1969) and Merton (1971) without pension insurance present. A random time horizon simply does not alter this result.

The formula (69) basically tells us that when prices of stocks increase, it is optimal to sell, and when prices fall it is optimal to buy, provided volatilities do not change. From an insurance perspective companies are often led to do the opposite, as we have mentioned before, which is of course unsatisfactory. However, in times of crises the volatility of the stock market index typically increases, in which case the optimal fraction in the index goes down.

In the formula (69) it is customary that γ is interpreted as relative risk aversion, and $1/\gamma$ is relative risk tolerance. The EIS-interpretation of $\psi = 1/\gamma$ does not appear relevant in this connection.

One objection to result (69) is that the optimal strategy does not depend upon the investor's horizon. This is against empirical evidence, and also against the typical recommendations of portfolio managers and insurance companies. The typical advice is that as the horizon gets shorter, the investor should gradually go out of equities, and thus take on less financial risk. This issue we present a short discussion of in the next section.

Another objection about this model is that under our assumptions about deterministic σ_t, ν_t and r_t this model implies that the volatility of the consumption growth rate σ_c is equal the volatility σ_W of the return rate on the wealth portfolio W_t. Thus, from Table 1 we notice that the market portfolio can not be a proxy for the wealth portfolio under these assumptions.

12 The horizon problem

In this part we examine the effect of horizon and wealth on portfolio choice. We assume that the felicity index $u(x, t)$ satisfies the following

$$u(x,t) = \begin{cases} \frac{1}{1-\gamma(t)} x^{(1-\gamma(t))} e^{-\delta t}, & \text{if } \gamma(t) \neq 1; \\ \ln(x) e^{-\delta t}, & \text{if } \gamma(t) = 1. \end{cases} \tag{70}$$

where $\gamma : [0, \tau) \to R_+$ is a continuous and strictly positive function of time. Notice that in this case $u(x, t)$ is not time and state separable, but this is the only relaxation of the standard assumptions that is done. Using this assumption, Aase (2009) shows that under this assumption the optimal fractions in the risky assets are

$$\varphi(t) = \frac{1}{\gamma(\tilde{t}_t)}(\sigma\sigma')^{-1}\nu, \tag{71}$$

where $\tilde{t}_t$ is an $\mathcal{F}_t-$measurable random time satisfying $\tilde{t}_t \in (t,\tau)$. It is determined at each time t by the equation

$$\gamma(\tilde{t}_t) = \frac{\int_t^\tau g(s,t)ds}{\int_t^\tau g(s,t)\frac{1}{\gamma(s)}ds} := \frac{W_t}{Z_t}. \tag{72}$$

Here W_t is the agent's optimal wealth at time t, given by equation

$$W_t = \int_t^\tau \left(\lambda e^{\delta s}\right)^{-\frac{1}{\gamma(s)}} \pi_t^{-\frac{1}{\gamma(s)}} \exp\Big\{-\Big(r + \frac{1}{2}\frac{1}{\gamma(s)}\eta'\cdot\eta\Big)\Big(1 - \frac{1}{\gamma(s)}\Big)\big(s-t\big)\Big\}\frac{l(x+s)}{l(x+t)}ds. \tag{73}$$

Notice that when the function $\gamma(t) \equiv \gamma$, then the wealth in this equation becomes the same as the wealth in (68), as the case should be. Clearly the quantity Z_t can be computed from the expression for W_t in (73) and the function $\gamma(t)$.

The consequences of this result are several, and the above reference gives the details. Here we only point out that if the risk aversion function $\gamma(t)$ is increasing with time, this result implies that individuals should invest more in the risky asset when they have a longer horizon, i.e., when they are young, and gradually move into bonds as they grow older. This is then in agreement with both advice from investment professionals, and with empirical studies of actual behavior.

It seems natural, with this assumption, that the investor should pick some average time in the remaining horizon when deciding on today's portfolio choice.

The horizon problem should, perhaps, be formulated in the mean reverting setting of Section 6, since one concern is what happens when retirement takes place in a slump. The technical side of this problem has been considered in Benth and Karlsen (2005), where the result presented is rather complicated, but the optimal ratio in the risky asset depends on the horizon. Another situation is when, in the the standard model, there is a bequest utility function v different from u. Time dependence can then arise in the optimal portfolio solution as well.

13 A second portfolio choice puzzle

In connection with the optimal portfolio choice result (69), there is also another empirical puzzle. Again we refer to the study of Mehra and Prescott (1985) of the US-economy for the period of 1889-1978, where the data are summarized in Table 1.

Based on the conventional, pure demand theory of Section 11, by assuming a relative risk aversion of around two, the optimal fraction in equity is 119% follows from the standard formula (69), using the summary statistics of Table 1, and assuming one single risky asset, the index itself. In contrast, depending upon estimates, the typical household holds between 6% to 20% in equity. Conditional on participating in the stock market, this number increases to about 40% in financial assets.

One could object to this that the conventional model is consistent with a value for γ around 26 only. Using this value instead, the optimal fraction in equity is down to around 7%, which in isolation is reasonable. However, such a high value for the relative risk aversion is considered implausible, as we have discussed before.

This is a problem where the recursive model gives much more reasonable results, see Aase (2014a).

14 Longevity and cohort risk.

We round off by discussing some issues that do not directly come as a result of the above analysis, but which are related to problems commonly discussed in connection with pensions.

In comparing longevity risk with cohort risk, it is tempting to dismiss the latter as not being of such fundamental importance as the former. By cohort risk is meant that some periods have larger numbers of retired people than other periods. This is a transient phenomenon that will eventually pass away, and not a structural one, as longevity risk. Of course, when these two risks materialize at the same time, this causes extra problems for any nation's welfare programs. This seems to be the case in many western countries when the large broods borne right after World War II become pensioners. In addition these cohorts tend to live longer than the generations before them.

In some countries the actuarial tables are modified every year, like in Canada, in other countries the same tables as were constructed in 1963 were still used in 2009, like in Norway. The theory in this paper assumes that the tables capture the real mortality risk, and pooling works so that there is no economic risk premium associated with mortality. As long as the proper measures have been taken regarding reserving for longevity risk, there should be few problems for the private insurance industry with respect to either of these two types of risk.

For government welfare programs, the situation is of course different. Many developed countries have a social security system that pays a basic pension to its citizens. This is usually independent of what the individuals have arranged in terms of pensions from the insurance industry. In Norway, for example, the country that I know best, the government pensions are determined by the principle of "pay as you go". For those only acquainted with the premium reserving of private or mutual insurance companies, this may not look like a sound principle. In the parliament (Stortinget) the politicians determine a basic amount each year, called one G upon which the pensions are based. In 2010 the size of $G =$ NoK 75.641, corresponding to USD 13.000. The more registered work effort an individual has put in, and the higher the salary, the higher the pension. Consider the incentives: By and large this arrangement means that the daughters and sons of the beneficiaries determine the benefits. Thus the "weak" part - the pensioners - seem protected, or at least, they get what they "deserve". Second, what about economic sustainability? Since all pensions are determined from the basic amount G, by making this amount state dependent, matters can be arranged such that the nation each year pays the pensions it can afford. As we have mentioned earlier, here nations are in addition able to carry out some sort of time diversification on the aggregate level.

In practice, to set G lower one year than the previous year may require a great deal of political determination and courage, which means that the system represents no guarantee that the nation will not consume beyond its means. Here rules rather than discretion may be the solution.

In addition to this basic pension from the government, and possible private pensions with the insurance industry, in many countries there are pensions also from the employers. These collective pensions are usually arranged between the employers and private insurers. The pensions depend upon how long an employee has been with the company, what the salary has been, and the premium reserve moves with the worker as he or she changes jobs.

The two types of risk, longevity risk and cohort risk, are problematic for governments' welfare schemes. One solution has been pointed out in a recent report[8]. By increasing the pensionable age by a few years, the projected increase in the state's pension expenses may be mitigated. In particular this report claims that by increasing the pension age by two years, this increases the state's income of about four per cent of GDP for the case of Norway. For

[8]`http://www.dn.no/forsiden/borsMarked/article2029034.ece`

an average working period of 40 years, an increase of two years means that the total work effort is society has increased by five per cent. In other words, society can become five per cent richer if people work two more years.

This suggestion has of course its weaknesses, since for once it "assumes away" unemployment, which is not negligible in many western countries. It is therefore also likely to be controversial. That it is politically difficult, we know from protests and demonstrations in 2010 and later in countries like Greece, Ireland, France, Portugal, Spain, etc. However, it is no secret that some countries seem to have more "slack" than others. As an illustration, in Table 2 is shown the employment frequency for people between 60 and 64 years for a number of European countries and the USA. It starts at about 7% in Austria, goes via 40% in the USA and ends with 58% in Sweden and Norway.

Freq.	7	12	12	21	22	31	35	40	43	58	58
Nation	Au	Fr	It	Sp	Ger	Gree	Den	US	UK	Swe	Nor

Table 2: Employment frequency in per cent, 60-64 years. Source: Eurostat.

The official and the real pension age also vary across the European countries, highest in Iceland with 67 and 66 years, and lowest in France with 60 and 59 years, respectively. The problems with longevity and cohort risk are thus seen to have both macro, public, and political economic perspectives.

15 Summary, discussion and extensions

The life cycle model is analyzed in two steps; first with only a credit market and mortality risk, then with a securities market added. The analysis provides an optimal demand theory from the point of view of the consumers, who are also potential life and pension insurance customers. In this model optimal insurance contracts are derived, assuming they exist, which we then compare to real contracts. We have derived several conclusions from this model, some with more predictive, or normative power than others, which we now summarize.

The first result was related to the optimal consumption path in the situation with only a credit market. When there is life time uncertainty, the optimal consumption paths are shown to be crucially dependent on the impatience rate. The impatient consumer ($\delta > r$) must always look forward to an ever decreasing optimal consumption, since $dc_t^*/dt = (r - \delta)T(c_t^*)$.

The patient agent ($\delta < r$), on the other hand, can look forward to an ever increasing optimal consumption.

While this gives an interesting and intuitive interpretation of the impatience rate δ, it is not likely to give reliable predictions. With a securities market included this property is diluted, by both a new addition to the drift term and a diffusion term. In particular the latter will dictate consumption paths to deviate from the simple, deterministic description just given. This opens up for interpersonal comparisons of consumption behavior at the same time in agents' life cycles. Impatience is more naturally discussed in terms of expectations when a stock market is present, in which case the market-price-of-risk, risk aversion and time preference (ψ^{-1}) must all be taken into account when characterizing this property.

The optimal pensions contain an additional random function when a stock market is included. This function is reciprocal to the state price density, a fact which was found to have several interesting implications. In particular the optimal pensions are found to be positively correlated with the economy in the sense that when stock prices are high, the pensions are also high, and vice versa. In the conventional model this a quite natural property, in particular for the aggregate economy, since such a consumption pattern is consistent with what the economy can deliver.

This inspired a discussion of what insurance companies can do to offer more reasonable pension and life insurance contracts to the public. We indicated that by making reserves, pensions can be offered that smooth the individuals life time consumption, which is clearly desirable in general, for any reasonable set of preferences. The reason that life insurance companies can offer such smoothing, is that they can time diversify in the financial markets, something individual customers can not manage quite that well.

We have a simple demonstration of the advantages of pooling with regard to pensions. It is shown that, with the same economic resources, the optimal yearly consumption is strictly larger with pooling, than without. This shows the mutuality idea is fruitful, a fact that is worth a reminder, in particular since we live in a time of individualism, seemingly picturing a world in which we are solely responsible for our own successes and failures.

Optimal life insurance, where the insured amount is endogenously determined, is analyzed, and its properties are found reasonable. Like pension insurance, also the insured amounts in life insurance are co-cyclical with the economy. This can be mitigated by the life insurance industry, just as for pensions.

It should be pointed out that we know little about the specification of the utility function v, when it serves a bequest motive, as compared to u for the standard model. Life insurance is an important financial tool for controlling inter-personal transfers, which necessitates references to the theory of transfers (like e.g., Bernheim, Schleifer and Summers (1985)). We show that if the insured amount is to be countercyclical to the economy, and thus be a bona fide insurance against tough times for the beneficiaries, this requires *risk proclivity* of the bequest function v. This effectively rules out this possibility. A countercyclical insured amount appears desirable in a finance setting, but risk proclivity does not. This is where the insurance industry can improve welfare.

We compared our results to both actuarial theory and insurance practice. With regard to pensions it was found that defined contribution plans are most in line with the optimal contracts found in this paper for the conventional model. However, consumption smoothing over the life cycle is preferred by all consumers, due to their time preference. Our conventional model is based on a representative customer, who, when calibrated to data, happens to have low substitution elasticity and high risk aversion and time preference. Although we do not want to draw too many conclusions directly from this, it can be pointed out that such an individual is likely to prefer to purchase a pension contract with guarantees.

We have argued that insurance companies should be especially well suited to take on market risk, since they normally have a long term perspective. This would enable them to *realize* the risk premiums in the market in the long run, which after all are time averages. This is *time diversification.*

If an insurance company only offers defined contribution pension plans, there is virtually no financial risk involved in this line of business, and as a consequence this company can only expect to earn about the risk free rate in equilibrium, in the long run. Such a return on the company's operations is unlikely to meet the requirements of its owners. However, the consequences for a privately owned insurance corporation is that equity can then be set low. With risk subsidies from, say the government, the average cost of capital is low. By the leverage effect debt has on total returns, the expected return on equity can then still be high.

Much has been written about the recent financial crises of 2007-9. One criticism of the financial industry that has been put forth is that the financial firms were eager to collect fees for their services, by inventing all kinds of products that were difficult to understand for ordinary customers. Such fees can not be directly considered as a compensation for risks, but were one basis for the profits in the industry. As long as prices went up, this seemed to work, but as soon as confidence in the system started to fail, the collapse came partly as a consequence of failed risk management, among other factors. However, this is not the major criticism of the industry - it is its ability to leave the downside risk to the taxpayers.

Financial firms trading in derivatives may access unbounded liability exposures and are granted limited liability. Under such circumstances an all equity firm holds a call option, whereby it receives a free option to put losses back to the taxpayers (e.g., Eberlein and Madan (2010)). In such a situation increasing volatility increases the value of both assets and the liabilities, thereby creating perverse incentives.

With defined contribution products, the insurers' equity can be kept low, but the return on equity should, at least in principle, only be high provided the insurers are clever in collection fees from the customers, since there is virtually no financial risk involved. If the products are largely standardized, competition should bring down these fees, and also the profit margins for the insurers. For this reason insurers are are sometimes ingenious in tailor making products to customers, where terms are opaque and difficult to compare.

In some countries there are state guarantees issued for individual pensions. As with banks, where the government has a stake because it insures deposits, the reason is to preserve the stability of the financial system, which is important to preserving the stability of the economy. If an insurance company gets into a situation of distress, the government may have to come in to honor its commitments to the insurance customers, which can be done by conservatorship. Because of the importance of thrust between the population and the life insurance industry, it is more common that life insurance companies in distress are taken over by other companies in the industry. In the 2007-09 crisis, in the US the government chose to provide funds to the financial firms with virtually "no strings attached". This may distort both risk management in the future, as well as proper pricing of the products.

Related is the desire to keep equity low, especially for institutions that are "too big to fail". By the Miller and Modigliani (M&M) theorems the value of the firms (like insurance companies and banks) should be independent of the capital structure so long as the investments are unaffected[9]. Since banks may be bailed out by the government, the cost of capital can be low in these sectors. The M&M-theory predicts that the cost of capital is a constant function of leverage. In practice, for an industrial firm the cost of capital is U-shaped as a function of the debt ratio. At the beginning it falls because of the tax advantage of debt, for larger value of the debt ratio it increases because of bankruptcy costs. This trade-off is typically broken for financial firms with an implicit government risk subsidy. Focusing on banks, the low average cost of capital is due to the government's role in the

[9]abstracting from bankruptcy costs, taxes, regulations, and agency problems

case of bankruptcy. By keeping equity low, the owners can obtain high returns on equity, not necessarily because of good risk taking, but rather due to the risk subsidy granted by government, combined with the leverage effect of debt on the expected rate of return on equity.

The low level of equity held prior to the 2007-crisis, together with a previous deregulation, also tempted some to higher risk taking, which meant that the investment profile changed as a consequence of the special capital structure in the banking industry. This is moral hazard, and certainly brings us outside the M&M framework. Prior to the financial crisis this behavior increased the risk (unnecessarily) for all the agents involved, except the creditors, the result of which is known by now. There are also elements of moral hazard in the relationship between owners and management. Many of the same features are present also in the life insurance industry, although to a lesser extent.

It is essential that the financial industry and the population at large learns from this, so that future crises become less severe. In order for the relevant requirement on equity and reserves to be appropriate, both incentives must be aligned with societal goals, and governments must get in place a proper regulatory regime that works. The 'too big to fail' doctrine must be broken, and the creditors must be forced to take potential losses. This may downgrade the debt of banks, but will motivate banks to increase their equity ratios.

Finally the paper discusses optimal portfolio choice strategies. This culminates with the formula (69), characterizing the optimal plan in the context of pure demand theory. When applied to market data this formula overestimates ordinary consumers' exposures to risky securities. As with all simple formulas, there are pros and cons. The advantage is the simple logic this formula conveys, the drawback is that it is framed in a very simple model of a complete, frictionless financial market, which is, perhaps sometimes taken too literally. One particular assumption about this market is that the investment opportunity set is constant. (The recursive model turns out to do much better in this regard, and can better explain observed data.)

Another weakness with the theory of optimal portfolio choice is related to the "horizon problem". Here we make a deviation from the additive and separable preference representation: We relax the separability of state and time in the felicity index $u(x, t)$ in the standard model. This is unusual in economic models. Nevertheless, it has the potential to explain observed behavior, namely that as investors grow older, they invest a larger proportion of their wealth in government bonds.

If we take into account also the supply side of the economy, and for example study Pareto optimal contracts, this may give a clearer picture of some of the problems discussed. We know that such contracts are "smooth" at each time, unless there are frictions of some kinds, and including the supply side of the economy may give contracts different from what follows by demand theory alone.

We observed that the growth rate of aggregate consumption in society has low estimated volatility when calibrated to data, with a relatively high estimate for the growth rate. This is not consistent with the model, hence the consumption puzzles of Section 5.

This general discussion revealed several weaknesses with the additive and separable framework of von Neumann and Morgenstern expected utility in a *temporal* context. One reason singled out is that the risk preference has two different, and sometimes conflicting interpretations in this model. We have therefore introduced recursive utility in a companion paper, containing a separation of consumption substitution from risk aversion. This gives a better explanation of market and consumption data, which opens up for several new interpretations.

Finally we presented some comments on longevity risk and cohort risk, and concluded that these problems are, perhaps, best analyzed in the perspective of macro, public, and political economics.

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